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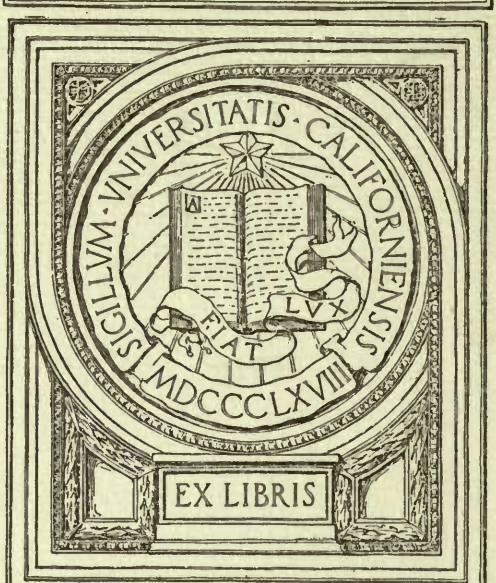
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RETAINING WALLS

BASED ENTIRELY ON THE THEORY OF FRICTION

ILLUSTRATED WITH 62 DIAGRAMS

BY

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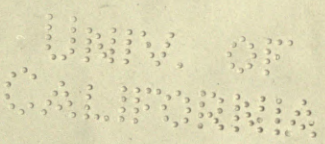

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PREFACE

The object of this treatise is to establish a general theory whereby we will be permitted to value the pressures received by a plane cutting an homogeneous unbounded mass in equilibrium, be this coherent, incoherent or liquid.

Referring to the group of coherent solids, up to the present no writer has investigated as to whether a plane cutting an homogeneous and strongly coherent solid, receives any pressure from that solid.

For the study in this same sense, of the other 2 groups of incoherent and liquid bodies, there are existing theories, but as far as I am informed none of them have proved satisfactory. One of these, the best known, and which possesses most logic, is that of the well known english writer, W. J. Rankine, but not even that has escaped serious criticism. In fact, M. A. Flamant, (Inspecteur General des Ponts et Chaussées) in his work on "Stabilité des Constructions" commenting upon Rankines theory states that experimentally it has been proved that said theory is erroneous when applied to incoherent bodies.

He states that the experiments made in England by Darwin and Baker, and in France by Mr. Gobin, have given lower results to those obtained by applying the theory, such differences in some cases being nearly one half.

Many other writers have dealt with the point, and established likenesses between the pressure excercised by incoherent and liquid bodies, but up to the present said likenesses have not been set down in satisfactory formulae.

The theory set forth in this book differs from all existing ones, from some in detail and from others fundamentally. Although it is noticeably in accordance with Rankine's as regards the principles on which it is founded, the results arrived at by its use are notably different, Taking the others generally, it disagrees with them both in fundamentals and in the results obtained.

The majority of writers, in order to arrive at the value of the pressure excercised by an incoherent solid on a retaining wall, start from the following basis:

To begin with, they suppose the existence of the wall; and assign an infinitesimal movement of translation parallel to itself, due to which the earth that is behind the wall, tries to fill the vacuum produced by the translation of same, thus causing, in the space between the earth and the back surface of the wall a friction that allows us to establish the direction and magnitude of the maximum pressure against same.

The above statements may be correct scientifically considered, but there is at least one missing link in the reasoning, and that is, one which will establish that the static force that actuates on the wall in equilibrium is the same in direction and magnitude to the one that actuates on the wall once the movement is iniciated.

Several other theories exist but the generality of same are derived from Poncelet's and Rankine's, referring which we have already given an outline.

The theory here laid down is based on the laws of friction and is limited to valueing the static pressures on a plane cutting an homogeneous unbounded solid in equilibrium, this, in practice, being substituted by a retaining wall.

As regards the algebraic formulae used to value the pressures; these are general to the 3 groups of bodies, with the only condition of in each case using the corresponding coefficient thus: should we be working with coherent bodies, in the formulae will figure the coefficient of resistance to the shearing force the coefficient of friction and of unit weight; for incoherent bodies only the coefficients for frition and unit

weight will figure and lastly for liquids only the unit weight, in this case, friction being zero.

In calculating the retaining walls, be these for incoherents or liquids, a theory has been established whereby, over and above the known conditions that are required for the stability of a wall is included the resistance to shear, a security factor of real importance.

To conclude I may state that I am quite aware of the great risk of publishing this book. The question herein dealt with, has been treated by such personalities as Colomb, Prony, Poncelet, Rankine, etc., etc., and even whilst the results here arrived at, differ considerably from those obtained by the above mentioned writers, I have not withheld publication as I am wholly convinced that my theory is logical.

This certainty does not arise out of any subjective conception, but is the result of a very careful and full study of the question. The aspect which I have taken for this study is so natural, the algebraic formulae arrived at for finding the values, so simple, and in each case respond in such a clear and general manner that I must necessarily agree that same are good. Lastly, the very fact that we are able, through quite a new channel, to establish the fundamental formulae of hydrostatics, dissipated any doubts I might have had respecting the logic of the theory I am herein expounding.

The author desires to express his thanks to his friend R. T. Mulleady B. Sc. Eng. for his generous help in the publication of this book.

The author will also be grateful for the notification of any error that may be found throughout this work,

PEDRO J. DOZAL.

Buenos Aires, Noviembre 1918.

ERRATA

In page 16, paragraph 11, line 4^{ht} from the bottom reads N T, it should read M T.

In page 56, paragraph 38, in 5^{ht} equation the term — $x^2 \tan \varphi$ should read — $x^2 \sin \varphi$.

In page 62, paragraph 43, in equation $\frac{d\ (L \cos \alpha)}{d\ \omega}$ the term $\omega^4 C$ should be — ω^4 C.

In page 80, last equation denominator $1 - 2\ \omega H\ 4\ \omega^2$ should read

$$1 - 2\ _{\omega}H + \omega^2$$

In page 88, top equation where reads 9 tan 2 φ should be 9 $\tan^2 \varphi$.

In page 103, first line under figure 41, it reads prism R M Q. should read prism R M O.

In page 124, line 9, from the bottom reads (a), it should read (7).

Page 129, the heading of paragraph 2 reads Incoherent Bodies, should read «Coherent Bodies».

RETAINING WALLS

CHAPTER I

PRELIMINARY DATA

Friction

1—Our study is based *(entirely)* on the friction between the substances, whose laws may be enunciated as follows:

Law I. «*When two bodies are in contact, the direction of the friction on one of them at its point of contact is opposite to the direction in which this point of contact would commence to move*».

Law II. «*The magnitude of the friction is, when there is equilibrium, just sufficient to prevent the body from moving*».

The above laws hold good in general, but the amount of friction that can be exerted is limited, and equilibrium is sometimes on the point of being destroyed and motion often ensues.

Limiting Friction—Definition. «*When one body is just on the point of sliding upon another body, the equilibrium is said to be limiting, and the friction then exerted is called limiting friction*».

The direction of the limiting friction is given by **Law I.** Its magnitude is given by the three following laws:

Law III. «*The magnitude of the limiting friction always bears a constant ratio to the normal reaction, and this ratio depends only on the substances of which the bodies are composed*».

Law IV. «*The limiting friction is independent of the extent and shape of the surface of contact, so long as the normal reaction is unaltered*».

Law V. «*When motion ensues, by one body sliding over the other, the direction of friction is opposite to the dirction of motion; the mag-*

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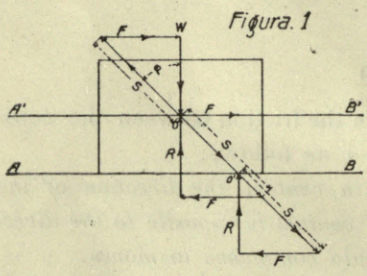
nitude of the friction is independent of the velocity, but the ratio of the friction to the normal reaction is slightly less than when the body is at rest and just on the point of motion».

2—Coefficient of Friction.—The constant ratio of the limiting friction to the normal pressure is called the coefficient of friction and is generally denoted by μ: hence if F be the friction and R the normal pressure between the two bodies, when equilibrium is on the point of being destroyed, we have $\frac{F}{R} = \mu$ and hence $F = \mu R$.

The values of μ are widely different for different pairs of substances in contact, and this value is always less than 1.

Angle of Friction—Let (Fig. 1) represent a body of weight W kilos resting on a horizontal table A B. The reaction of the table on the body is R. Let us apply a horizontal force F at its *c. g.* in the direction shown in the figure and let us suppose that this force has such a value that equilibrium is on the point of being broken.

Figura 1



The only horizontal force preventing motion is the force of friction F. This force by (**Law III**) is equal to μ R.

Resolving horizontally and vertically, we have that:

$$F = \mu R$$

$$R = W$$

$$\text{Therefore } F = \mu W$$

The resultant of the reaction and force F is S and the angle formed by the resultant S with the vertical is called the angle of friction.

In the triangle of forces *O W F* we have that:

$$\frac{\mu R}{R} = tan\ \varphi$$

Therefore:

$$\mu = tan\ \varphi$$

Hence we see that the coefficient of friction is equal to the tangent of the angle of friction.

3—Equilibrium on a rough inclined plane.

Let a body of weight W rest on a horizontal plane A B (fig. 2) let R be the normal reaction of the plane. In this position there is no horizontal force acting on the body so the force of friction does not come into play.

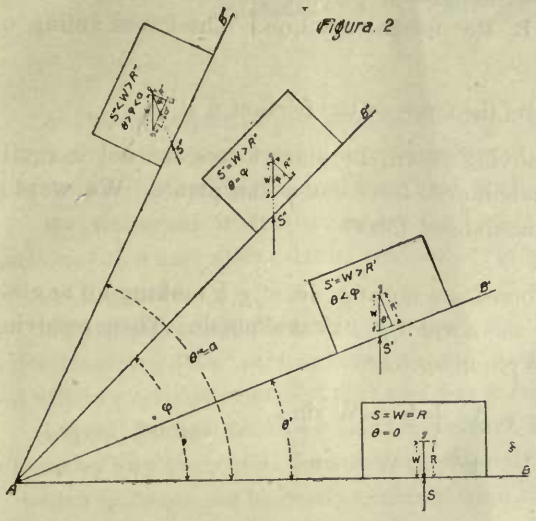


Figura 2

Now let body rest along plane A B′ making an angle θ′ less than the angle of friction. The body in this position will not move although the component of the weight of the body is pulling along the plane. The motion down the plane is prevented by the force of friction μ R and in this position both forces are balancing each other. If we increase the inclination of the plane up to φ, *i. e.* up to the angle of friction, we will still have the body at rest but on the point of moving, *i. e.* as soon as we increase the inclination of the plane by any amount, motion will ensue. The body in position A B″ is just on the point of moving and the force of friction μ R″ is the limiting friction.

Let us find the relation between the coefficient of friction and the inclination of the plane.

The forces keeping the body in equilibrium are:

$$W,\ R''\ \text{and}\ \mu R''.$$

Resolving along plane we have:

$$(1)\quad \mu R'' = W \sin \varphi$$

Resolving normal to the plane we have:

$$(2)\quad R'' = W \cos \varphi$$

Dividing (1) by (2)

$$\mu = \tan \varphi$$

So when the inclination of the plane is equal to the angle of friction, the coefficient of friction is equal to the tangent of the angle. The angle of friction is called angle of repose.

Now suppose a body is placed on a rough plane, inclined to the horizon at an angle greater than the angle of friction.

In this case there will not be equilibrium and the body will slide down the plane. Let α be the inclination of plane to the horizon, W the weight of the body and R''' the normal reaction. The forces acting on the body are:

W, R''' and the force of the friction μ R'''.

Since the body is moving down the plane forces are not in equilibrium and there is an unbalanced force down the plane. We want to find the value of this «unbalanced force».

Let L be this force.

In the triamgle of forces $g\,a\,m$, at g draw $g\,b$ making an angle φ (angle of repose) with the normal $g\,a$ to the plane. Then resolving along and normal to the plane we have

$$\mu\, R''' + L = W \sin \alpha$$

$$R''' = W \cos \alpha$$

Therefore:

$$(1) \quad L = W (\sin \alpha - \cos \alpha \tan \varphi)$$

This force L which we shall call in future the «unbalanced force» is given in magnitude and direction by $b\,m$ in the triangle of forces. This force L will drag the body down the plane with a theoretically uniformed accelerated motion.

In equation (1) we see that the value of L is a function of α. since W is also a function of α.

If we increace α up to 90^0 L becomes in equation (1) equal to W i.e. the unbalanced force is equal to the weight of the body.

What we have said about a body resting on a rough inclined plane we may apply to a mass of earth, sand or any other incoherent substance resting on the ground. These substance will settle, making an angle with the horizon equal to the angle of repose.

CHAPTER II

ELEMENTARY PRINCIPLES

4—The first part of this book is divided up into three portions.

The first portion will treat on coherent substances, the second will deall with incoherent and the third with liquids.

By Coherent Bodies we understand all those solid substances the portions of which offer certain resistance to their separation from the rest of the mass.

By Incoherent Bodies we understand all those solid substances in a desintegrated state such sand, earth, etc. that can be separated without offering resistance except that one due to friction.

Liquid Bodies are those substances such that their shapes can be altered by any tangential force however small, if applied long enough, of which portions can be easily separate from the rest of the mass and between different portions of which there is no tangential force of the nature of friction.

We shall consider, unless stated, all these kind of sustances forming unconfined homogeneous masses in a state of equilibrium.

Pressures at different points of an homogeneous body in equilibrium

5. Consider an unbounded and homogeneous mas of a body, be it coherent, incoherent, or liquid, in a state of equilibrium. Imagine that mass cut through by a plane A B. Since the mas is in equilibrium the pressures on each side of the plane are equal.

Take now a similar mass but instead of being unbouded is limited at the top by a horizontal plane A B (fig. 3).

Consider a horizontal and thin strip of body $k\ l$ and within it an element J. Since this element is in equilibrium and at equal distance from free surface A B. the pressures on each side of the element are equal. The same reasoning may be applied to any other element within the strip $k\ l$ or in the plane determined by $k\ l$ and the direction perpendicular to the paper.

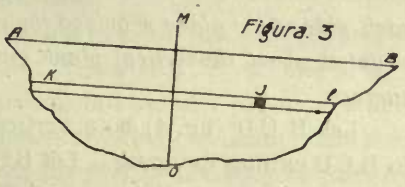


Figura 3

So we may conclude that *for any point on a horizontal plane inside a mass of a body bounded at the top for a horizontal plane the pressures round that point are equal.*

If the plane bounding the mass is inclined, the above statement is true for coherant and incoherent bodies but it fails in the case of liquids since a liquid surface in equilibrium is always horizontal.

Now consider a mass of a coherent or an incoherent body bounded at the top by an incline plane B C (fig. 4). In this case the pressures exerted on any plane *b c* within the body and parallel to the bounding plane B C will only be equal for elements symetrically placed on each side of the vertical plane B C D cutting the mass.

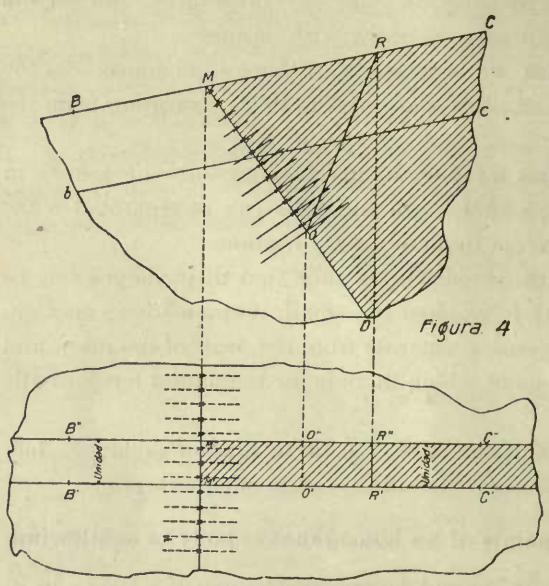


Figura 4

For an incoherent substance the inclination of surface B C bounding the mass cannot be greater than the angle of friction, and for a coherent body this inclination can be anything within the limits allowed by the coherence of the substance considered. Having that in our mind, and also considering that any line in an horizontal plane determines the maximun inclination of the plane we may state the following:

In any cutting plane parallel to the plane limiting the top of a homogeneous coherent, incoherent or liquid body the pressures are equal on each side of the plane provided the elements considered are symetrically situated about the vertical plane giving the maximum inclination of the surface.

Let B C D (fig. 4) be a vertical section given by a plane parallel to B C D cutting the mass. Let B C be the plane bounding the top and be perpendicular to the plane of the paper.

Consider two vertical parallel planes at a unit distance apart and also parallel to the maximum gradient of the plane B C D cutting the block and let B′ C′ and B″ C″ in plan be those planes. Consider a third plane M D perpendicular to the other two cutting the mass.

The pressures on each side of plane M D will be equal for any point on the plane an also for any point on any plane parallel to the one bounding the mass considered.

Now let us take out of the big mass the portion limited by the parallel planes B′ C′ and B″ C″ and the one perpendicular to them M D and let C M D be the vertical section considered. Let us study the pressures on this small mass cut out of the whole mass.

CONDITIONS

6. Before demonstrating the general formula for the thrust of *earth* against a wall it will be necessary to state some conditions within which we have based our study.

1.° We consider the bodies perfectly homogeneous and rigid.

2.° The coefficient of friction and ultimate shearing stress are constant no matter on which direction the forces are acting.

With reference to the first condition we may say that in nature there are not perfectly homogeneous and rigid bodies; but we must consider them so, in order to be able to deal with them in a reasonable and practical way.

CHAPTER III

NOMENCLATURE

7—General Mass is an unconfined homogeneous mass in a state of equilibrium bounded at the top by a plane.

Mass. We shall call in future by the name of mass the portion of the general mass separated by three planes B′ C′, B″ C″ and M′ M″, where B′ C′ and B″ C″ are two vertical and parallel planes at a unit distance apart and M′ M″ is perpendicular to both of them. The portion shown shaded in (fig. 4) encloses the part of the body referred as mass.

Paramento. Is the surface determined by the plane M D perpendicular to the plane of the paper.

Prism is the volume enclosed between the paramento, the two vertical planes B′C′ and B″C″ and the plane R O, i.e. it is a prism having for base the triangle M R O and for height the unity M′ M″.

Note: In all the demonstrations which follow the dimensions perpendicular to the plane of the paper will be considered as unity.

Angle α is the angle made by plane R O with the horizontal.

Plane of condition is the plane determined by R O perpendicular to the plane of the paper.

Standard Body is a body perfectly homogeneous with a coefficient of friction and an ultimate shearing stress constant for any direction.

Unbalanced Force. If on a plane having an inclination to the horizon greater than the angle of friction we place a mass of a body, this mass will be pulled down the incline by a force. This force is called the *unbalanced force.*

Angle of Repose (φ) or the maximum angle which any force acting upon any plane within the mass of a body can make with the normal to the plane.

Coefficient of Friction is the tangent of the angle of repose.

Unit Weight (γ) is the weight of the unit mass considered.

FIRST PART

PRESSURE ON A PLANE CUTTING AN HOMOGENEOUS MASS

COHERENT BODIES

Problem N.° I

8 «*Find the general equation for the unbalanced force*».

Let T M O H (fig. 5) be a mass of a standard coherent body bounded at the top by a horizontal plane M T and with a vertical paramento M O. Let H O T be the angle of repose φ and H O R the angle α made by the plane of condition with the horizonal.

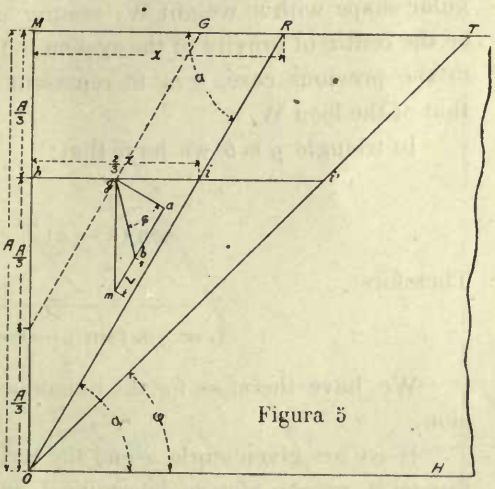


Figura 5

For prism T M O the component of the weight parallel to the plane O T is counterbalanced by the force of friction along the plane of repose, i. e. there is no force pulling along the incline.

For prism R M O we have the weight of the body W, the normal reaction R on plane O R, the force of friction μ R and the unbalanced force L.

Through el centre of gravity g of the prism R M O draw a vertical line $g\ m$ to represent the weight W to any scale and resolve this force in two components, one parallel to plane O R and another normal to it. Let $a\ m$ and $g\ a$ be those forces.

The angle $m\ g\ a$ between these forces is α and greater than the angle of repose. The component force $a\ m$ would be the only force pul-

ling along the plane of condition if friction were zero, but as friction is not zero, the resultant force dragging the mass down the incline is the difference between the component of the weight parallel to the plane and the force of friction. The amount of friction is given in the triangle of force by $a\,b$ and is obtained to the same scale drawing from point g, the line $g\,b$ making with $g\,a$ the angle of repose.

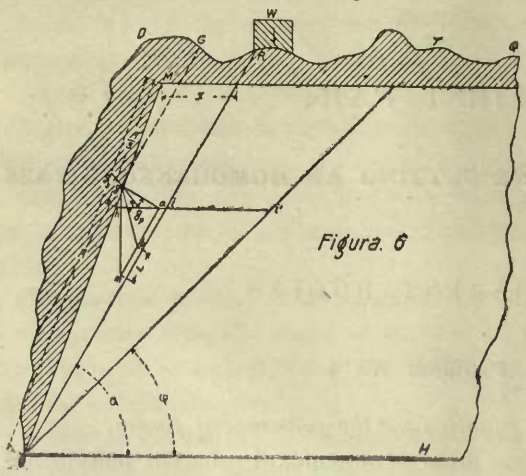


Figura. 6

This unbalanced force is given to scale by $b\,m$.

Consider now (fig. 6) the prism T D O of irregular shape with a weight W, resting on the top as shewn. Let g be the centre of gravity of the system. Draw the triangle of forces as in the previous case, $g\,m$ to represent the weight of the prism and that of the load W,

In triangle $g\,m\,b$ we have that:

$$\frac{L}{\text{sen}\,(\alpha - \varphi)} = \frac{gm}{\cos\,\varphi}$$

Therefore:

$$L = gm\,(\sin\,\alpha - \cos\,\alpha\,\tan\,\varphi)$$

We have therefore for the unbalanced force L, a general expression.

If we are given angle α and the weight of the prism and load resting on it, we can always determine L and consequently the component of L in any direction required. So the general equation for the unbalanced force is:

$$L = W\,(\sin\,\alpha - \cos\,\alpha\,\tan\,\varphi)$$

where W is total weight of prism and loads resting on it.

9 Let us now examine this equation and see how it varies.

Let T M O H be the mass considered (fig. 7), O T the plane of repose making with the horizontal an angle φ.

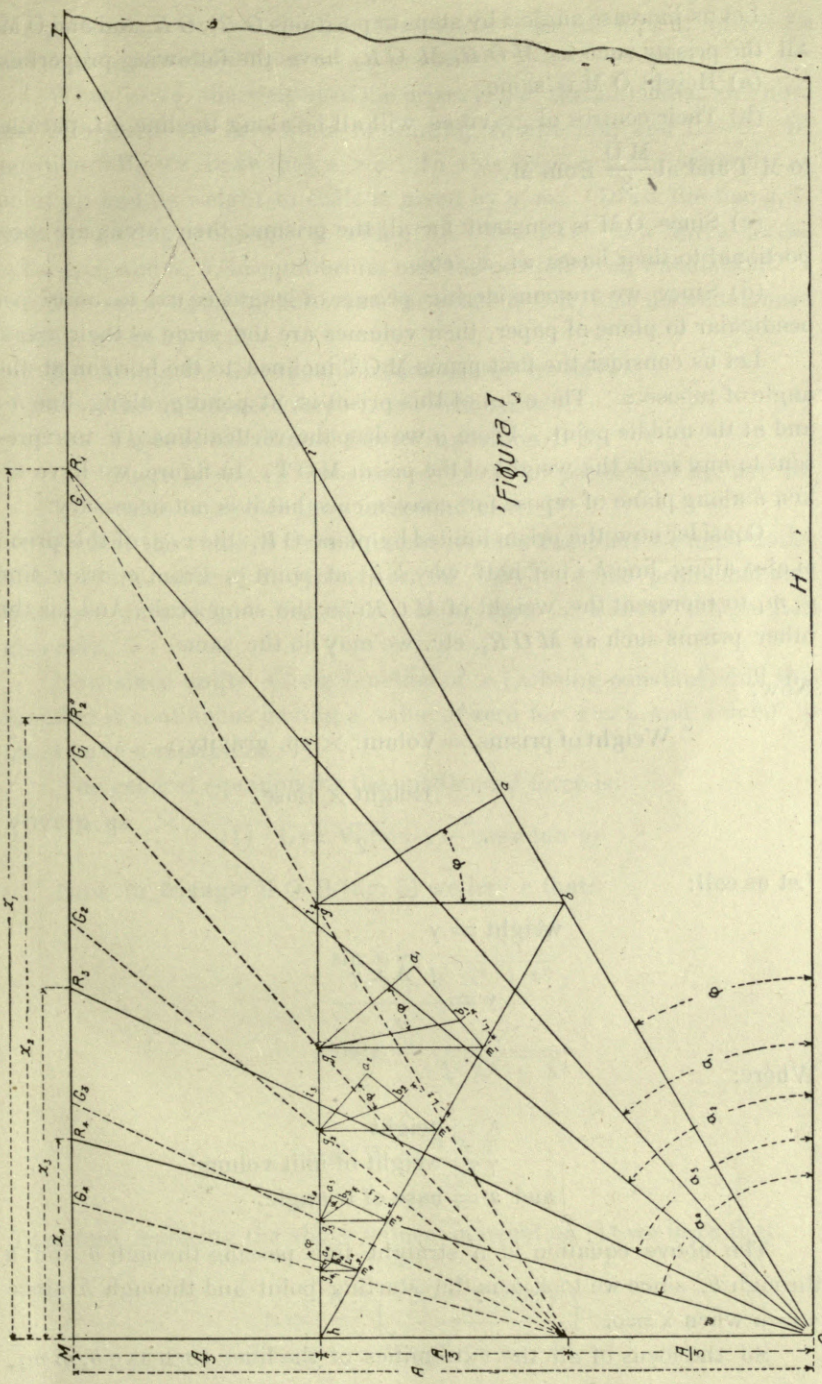


Figura 7

Let us increase angle α by steps to positions $O\ R_1$, $O\ R_2$ and and O M. All the prisms such as $M\ O\ R_1\ M\ O\ R_2$ have the following properties:

(a) Height O M is same.

(b) Their centres of gravities will all be along the line $h\ i$, parallel to M T and at $\frac{M\ O}{3}$ from M.

(c) Since O M is constant for all the prisms, their areas are proportional to their bases x_1, x_2, etc.

(d) Since we are considering prisms of lenght equal to unity per pendicular to plane of paper, their volumes are the same as their areas.

Let us consider the first prims M O T inclined to the horizon at the angle of repose φ. The c. g. of this prism is, at point g, along, line $h\ i$ and at the middle point. From g we drop the vertical line $g\ b$ to represent to any scale the weight of the prism M O T. In figure we have taken b along plane of repose for convenience but it is not necessary.

Consider now the prism limited by plane O R_1, the c. g. of this prism is also along line $h\ i$ buf half way $h\ i_1$, at point g_1 From g_1 draw line $g_1\ m_1$ to represent the weight of $M\ O\ R_1$ to the same scale. And for the other prisms such as $M\ O\ R_2$, etc. we may do the same.

Now:

$$\text{Weight of prisms} = \text{Volum.} \times \text{sp. gravity.}$$

$$= \frac{\text{Height} \times \text{Base}}{2} \times \text{sp. gravity}$$

Let us call:

$$\text{weight} = y$$

$$y = \frac{A \times \gamma}{2}$$

Where:

A = height

γ = weight of unit volume

and x = base of triangle.

The above equation is a straight line passing through b and h through b, since we took b as the starting point and through h since $y = o$ when $x = o$.

So the locus of all the extremities of the lines such as $g\ b$, $g_1\ m_1$,

$g_2\, m_2$, etc., is a straight line $h\, b$. Now let us examine how L varies as the angle α increases from $\alpha = \varphi$ to $\alpha = 90°$.

When $\alpha = \varphi$, the weight of the prism is $g\, b$, the tangential and normal component are in a state of limiting equilibrium and L = 0. In position O R_1 we have that $\alpha > \varphi$. In this case c. g. of prism is at point g_1, and its weight to scale is given by $g_1\, m_1$. Draw the line $g_1\, b_1$ from g_1 making an angle φ with $g_1\, a_1$. We have now three forces $g_1\, b_1$, $g_1\, a_1$ and $a_1\, b_1$ in equilibrium and the portion $b_1\, m_1$ unbalanced.

The same thing applies when the plane is O R_2 and the unbalanced force is given by $b_2\, m_2$.

Examining the three positions considered we have:

1° The unbalanced force L = 0 when $\alpha = \varphi$.

2° There is unbalanced force for position O R_1.

3° There is also an unbalanced force for position O R_2 but its value is greater than in second case.

So that the value of force L increases as α does, but we must notice also that the value of L depends on the weight of the prims and as α increases this weight disminishes being zero for $\alpha = 90°$ where L is also zero.

Now since angle α is a function of x (A being constant) and this function is continuous having a value of zero for $\alpha = \varphi$ and $\alpha = 90°$, it must have a maximum.

The general equation for the unbalanced force is:

$$(1) \quad L = W (\sin \alpha - \cos \alpha \tan \varphi)$$

Now in triangle M O R (fig. 5) we hav e that:

$$\sin \alpha = \frac{A}{\sqrt{A^2 + x^2}}$$

$$\cos \alpha = \frac{x}{\sqrt{A^2 + x^2}}$$

$$\text{and: } W = \frac{A \times \gamma}{2}$$

Therefore replacing the above values in equation (1) we have that:

$$L = \frac{A\ \gamma}{2} \left(\frac{A\, x - x^2 \tan \varphi}{\sqrt{A^2 + x^2}} \right)$$

For a maximum $\frac{d\,L}{d\,x}$ must be zero.

Now: $$\frac{d\,L}{d\,x} = \frac{A\gamma}{2}\left(\frac{A^3 - x^3 \tan\varphi - 2\,A^2\,x\,\tan\varphi}{(A^2 + x^2)\,\sqrt{A^2 + x^2}} \right)$$

Equating to zero we have that:

$$A^3 - x^3 \tan\varphi - 2\,A^2\,x\,\tan\varphi = 0$$

This is an equation of third degree whose roots are:

$$x_1 = A\left[\sqrt[3]{\frac{\cot\varphi}{2} + \sqrt{\frac{\cot^2\varphi}{4} + \frac{8}{27}}} + \right.$$

$$\left. + \sqrt[3]{\frac{\cot\varphi}{2} - \sqrt{\frac{\cot^2\varphi}{4} + \frac{8}{27}}} \right]$$

$$x_2 = A\left[k \sqrt[3]{\frac{\cot\varphi}{2} + \sqrt{\frac{\cot^2\varphi}{4} + \frac{8}{27}}} + \right.$$

$$\left. + k' \sqrt[3]{\frac{\cot\varphi}{2} - \sqrt{\frac{\cot^2\varphi}{4} + \frac{8}{27}}} \right]$$

$$x_3 = A\left[k' \sqrt[3]{\frac{\cot\varphi}{2} + \sqrt{\frac{\cot^2\varphi}{4} + \frac{8}{27}}} + \right.$$

$$\left. + k \sqrt[3]{\frac{\cot\varphi}{2} - \sqrt{\frac{\cot^2\varphi}{4} + \frac{8}{27}}} \right]$$

k and k′ are the imaginary cubic roots of the unity. Out of these three values only the first one is real.

Knowing distance x we find from triangle M O R that

$$\tan\alpha = \frac{A}{x}$$

therefore angle α can be obtained from tables and maximum value of the unbalanced force calculated from the general equation.

Value for the maximum horizontal and vertical components

10 The horizontal component of the unbalanced force is: L cos α. Now:

$$L \cos \alpha = W (\sin \alpha - \cos \alpha \tan \varphi) \cos \alpha$$

$$(2) \qquad L \cos \alpha = \frac{A \gamma}{2} \left(\frac{A x^2 - x^3 \tan \varphi}{A^2 + x^2} \right)$$

For a maximum $\frac{d (L \cos \alpha)}{d x}$ must be zero

We have that:

$$\frac{d (L \cos \alpha)}{d x} = \frac{A \gamma}{2} \left(\frac{2 A^3 x - 3 A^2 x^2 \tan \varphi - x^4 \tan \varphi}{(A^2 + x^2)^2} \right)$$

Therefore equating to zero we get that:

$$2 A^3 - 3 A^2 x \tan \varphi - x^3 \tan \varphi = 0.$$

This is an iquation of third degree whose roots are:

$$x_1 = A \left[\sqrt[3]{\cot \varphi + \sqrt{\cot^2 \varphi + 1}} + \sqrt[3]{\cot \varphi - \sqrt{\cot^2 \varphi + 1}} \right]$$

$$x_2 = A \left[k \sqrt[3]{\cot \varphi + \sqrt{\cot^2 \varphi + 1}} + k' \sqrt[3]{\cot \varphi - \sqrt{\cot^2 \varphi + 1}} \right]$$

$$x_3 = A \left[k' \sqrt[3]{\cot \varphi + \sqrt{\cot^2 \varphi + 1}} + k \sqrt[3]{\cot \varphi - \sqrt{\cot^2 \varphi + 1}} \right]$$

Out of these three values only the first one is real.

Knowing x we calculate valus of:

$$\tan \alpha = \frac{A}{x}$$

from triangle M O R and angle α can be obtained from tables and the maximum value of the horizontal component calculated from (2).

In a similar way the value of x that will give the maximum vertical component can be obtained:

$$L \sin \alpha = W (\sin \alpha - \cos \alpha \tan \varphi) \sin \alpha$$

$$(3) = \frac{A^2 \gamma}{2} \left(\frac{A x - x^2 \tan \varphi}{A^2 + x^2} \right)$$

Differenciating we have that:

$$\frac{d\,(L \sin \alpha)}{d\,x} = \frac{A^3\,\gamma}{2}\left(\frac{A^2 - x^2 - 2\,A\,x \tan \varphi}{(A^2 + x^2)^2}\right)$$

for a maximum $\frac{d\,(L \sin \alpha)}{d\,x}$ must be zero.

Therefore equating to zero we have that:

$$A^2 - x^2 - 2\,A\,x \tan \varphi = 0.$$

This is a quadritic equation roots are:

$$x_1 = A\left(\frac{1 - \sin \varphi}{\cos \varphi}\right)$$

$$x_2 = -A\left(\frac{1 + \sin \varphi}{\cos \varphi}\right)$$

Out of these two values the first one give us the maximun value of the vertical component, since this value makes the second differential coefficient of the force L sin α negative.

Therefore in triangle R M O we have that:

$$\tan \alpha = \frac{\cos \varphi}{1 - \sin \varphi}$$

and consequently angle α is obtained from tables and the maximun value of the vertical component for the unbalanced force can be calculated from equation (3).

SOLUTION OF EQUATIONS oF 3rd. AND 4th. DEGREE

11 We shall deal now with a question that will come to us pretty often in the course of this wortk. I am referring to the solutions of the equations of 3rd. and 4th. degree obtained by equating the differential coefficient to zero.

The most simple way of solving those equations is by trials.

Let T M O be the mass considered with a vertical paramento M O and a horizontal surface N T (fig. 8).

At 1/3 A from M draw the horizontal line *h i'* to meet line O T.

The centre of gravity of prism T M O is at point *g'* half-way along line *h i'*.

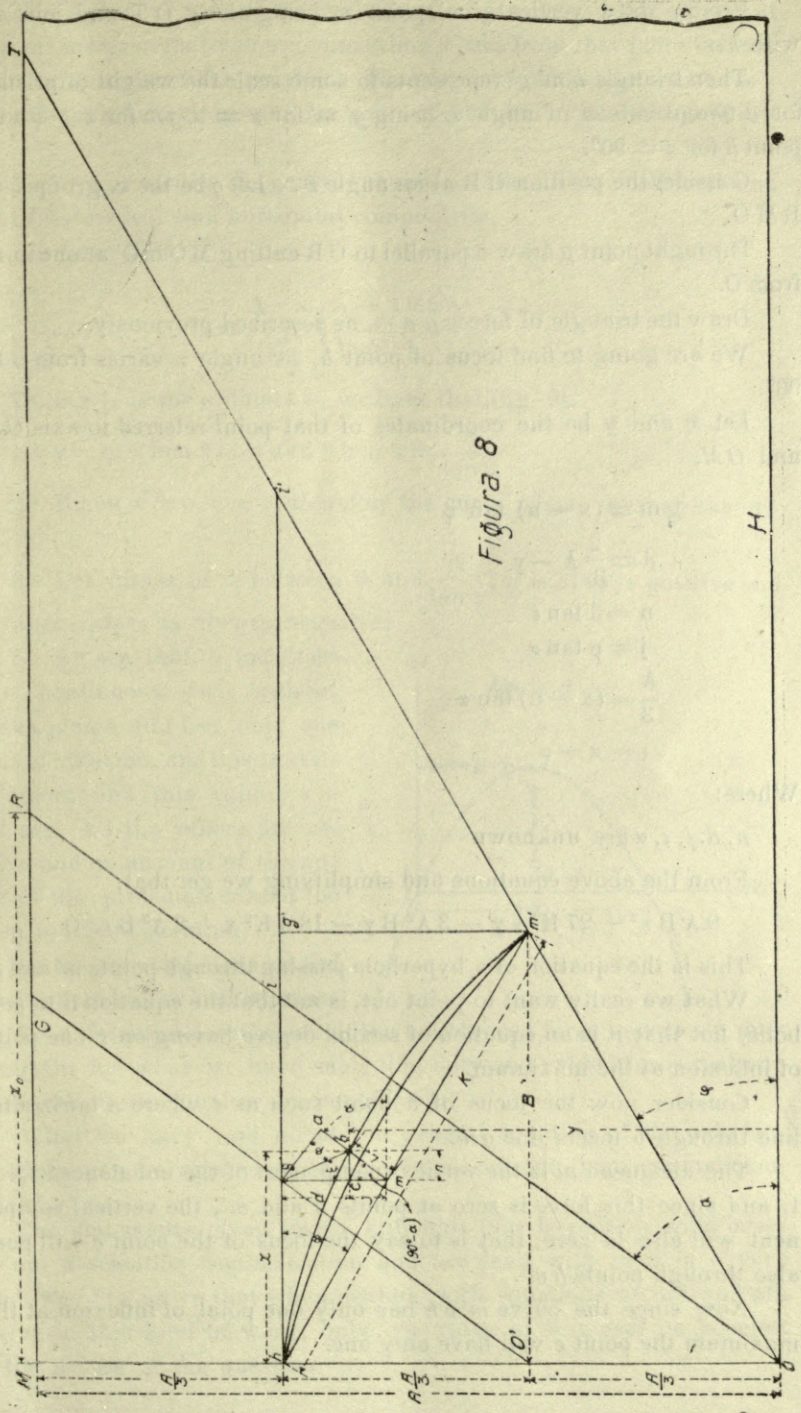


Figura. 8

From g' draw vertical $g'\,m'$, point m' being along O T and join m' with h.

Then triangle $h\,m'\,g'$ represents to some scale the weight of prisms for different values of angle α, being $g'\,m'$ for $\alpha = \varphi$, $g\,m$ for $\alpha > \varphi$ and point h for $\alpha = 90°$.

Consider the position O R at an angle α. Let g be the c. g. of prism R M O.

Throught point g draw a parallel to O R cutting M O in O' at one third from O.

Draw the triangle of forces $g\,a\,m$, as described previously.

We are going to find locus of point b, as angle α varies from φ to 90°.

Let x and y be the coordinates of that point referred to axis $O\,H$ and $O\,M$.

$$g\,m = (x - n)\,\tan\,\varphi$$

$$d = \frac{2}{3}\,A - y$$

$$n = d\,\tan\,\varepsilon$$

$$j = n\,\tan\,\alpha$$

$$\frac{A}{3} = (x - n)\,\tan\,\alpha$$

$$\varepsilon = \alpha - \varphi$$

Where:

n, d, j, ε, α are unknown

From the above equations and simplifying we get that,

$$9\,A\,B\,x^2 + 27\,K^2\,x\,y - 3\,A^2\,B\,y - 18\,A\,K^2\,x + 2\,A^3\,B = 0$$

This is the equation of a hyperbola passing through points m' and h.

What we really want to point out, is not that the equation is hyperbolic, but that it is an equation of second degree having only one point of inflexion at the maximum.

Consider now the locus of a point such as c where a horizontal line through b meets line $g\,m$.

The distance $c\,m$ is the vertical component of the unbalanced force L and since this force is zero at points h and m', the vertical component will also be zero, that is to say the locus of the point c will pass also through points $h\,m'$.

Now since the curve $m'\,b\,h$ has only one point of inflexion at the maximum the point c will have only one.

The distances such as $b\,c$ will therefore also have value of zero at point m' increase that value to a maximum and from that point decrease to value zero at the point h. Therefore the curve of the locus is a continuous one, having only one point of inflexion at its maximum value.

Now let us analise the general equation of the unbalanced force and that of its vertical and horizontal components.

We have that:

$$L = \frac{A\,\gamma\,x}{2}\left(\frac{A - x\tan\varphi}{\sqrt{A^2 + x^2}}\right)$$

Taking L as the ordinate y, we have that (fig. 9).

1.° $y = o$: when $x = o$ and when $x = \dfrac{A}{\tan\varphi}$

2.° When $x = o$, $y = o$, therefore the curve passes through the origin.

3.° For values of x between 0 and $\dfrac{A}{\tan\varphi}$, y is always positive and for other values is always negative.

So we see that, y increases in a continuous way, without sudden jumps and has only one point of inflexion, and this is at its maximum and this value is a *real one*. All the others are negative and on account of the nature of the problems cannot be taken into consideration,

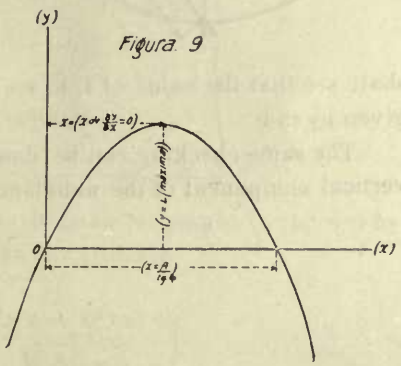


Figura 9

So in the equations of 3rd and 4th degrée obtained by equating the differential coeeffiicient of the unbalanced force to zero we may be certain for what we have said, that only one value of x is real and all the others are equal or are imaginaries.

What we have said about the general equation of the unbalanced force, can be applied to those of the horizontal and vertical components.

The statements given above, although they have very solid bases are not a scientific demonstration and are dealt with here in a very rapid way, to prove that when dealing with equations of 3rd and 4th degree in this kind of work, only one value of x is real on account of the nature of the question.

As a quick check for the angle of α which gives the maximum unbalanced force or the maximum horizontal and vertical component the student is advised to do the following graphic method.

12.—Graphic Method. Let $g\,m$ to any scale represent the total weight of the prism cut by plane of condition α obtained from calculations.

With $g\,m$ as diametre draw a circle as in (fig. 10). Throgh point m draw line $m\,a$ making with the horizontal the angle α calculated, and produce this line to cut circle at a, join a and g and through g draw line $g\,b$ making with $g\,a$ the angle of repose φ. Then $b\,m$ is the unbalaced force to scale and $c\,b$ and $c\,m$ the horizontal and vertical component of the force.

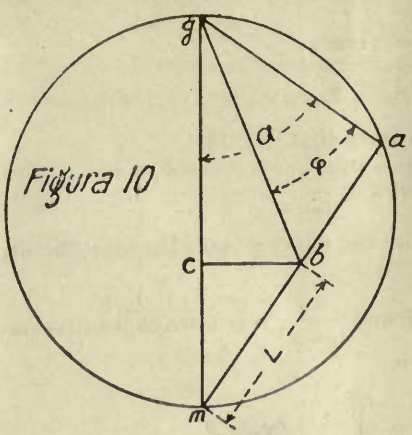


Figura 10

Now to check the result of the calculations do the same thing for an angle a little greater and a little less than α and we shall see that the value of L to scale in both cases is less than the one given by m b.

The same checking can be done for the maximum horizontal and vertical component of the unbalanced force.

CHAPTER IV

Problem N.° 2

13—*Given a mass* T M O H (fig. 11) *we require to find the plane of minimum resistance.*

Let O R be this plane.

By a plane of minimum resistance we mean that plane in wich the

ratio of the unbalanced force to the area is at its máximum, *i*, *e*. the plane in wich the max, shearing stress occurs. The longitudinal componet *a m* of the weight *g m* is pulling along plane O R. This force is count eracted by the force of friction given by *a b* and by the resistance to shearing of the material.

So we have: a m = a b + shearing force.

shearing force = a m — b a.

and this shearing force is the only force acting in opposition to the unbalanced force.

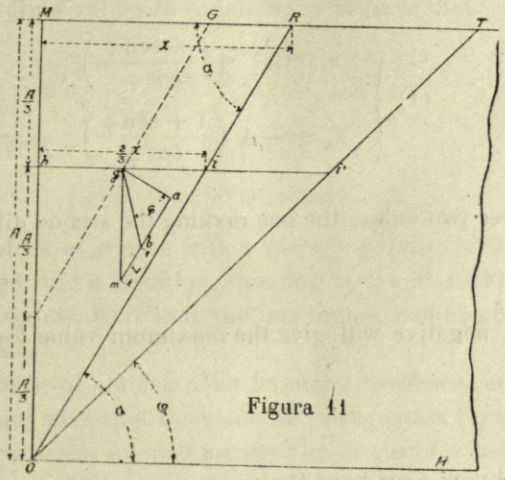


Figura 11

So if we find a plane, where this shearing resistance is opposed by a maximum unbalanced force we solve our problem.

$$\text{Now} \quad L = \frac{A\gamma}{2}\left(\frac{A\,x - x^2 \tan\varphi}{\sqrt{A^2 + x^2}}\right)$$

The sectional area is-

$$S = \sqrt{A^2 + x^2}$$

$$\text{Therefore shearing stress} \quad \frac{L}{S} = \frac{A\gamma}{2}\left(\frac{A\,x - x^2 \tan\varphi}{A^2 + x^2}\right)$$

$$\text{For a maximum} \quad \frac{d\left(\frac{L}{S}\right)}{dx} \text{must be zero}$$

Now;

$$\frac{d\left(\frac{L}{S}\right)}{dx} = \frac{A^2\gamma}{2}\left(\frac{A^2 - 2Ax\tan\varphi - x^2}{(A^2+x^2)^2}\right)$$

Therefore equating to zero we have:

$$A^2 - 2Ax\tan\varphi - x^2 = 0$$

This is quadratic equation whose values are:

$$(1)\quad \begin{cases} x_1 = A\left(\frac{1-\operatorname{sen}\varphi}{\cos\varphi}\right) \\ x_2 = -A\left(\frac{1+\operatorname{sen}\varphi}{\cos\varphi}\right) \end{cases}$$

Out of these two values the one making the second differential coefficient $\frac{d^2\left(\frac{L}{S}\right)}{dx^2}$ negative will give the maximum value for the shearing stress.

Since we know x we have that;

$$\tan\alpha = \frac{A}{x} = \frac{\cos\varphi}{1-\operatorname{sen}\varphi}$$

and therefore angle α can be found out from the tables.

From equation $x = A\left(\frac{1-\operatorname{sen}\varphi}{\cos\varphi}\right)$ we have that the ratio $\frac{x}{A}$ is constant for a given material, i. e. the planes of the minimum resistance will be parallel as the height A increases or diminishes.

Now the shearing strees is:

$$\Omega = \frac{L}{S}$$

substituting value of x from equation (1) and rèducing we have that;

$$(2)\ A = \frac{4\ \Omega \cos \varphi}{\gamma\,(1 - \text{sen}\ \varphi)}$$

This equation gives us at once the height A in terms of the shearing stress, angle of friction and specific gravity.

Example. Find the maximum height that a block of wrought iron of the shape shown in (fig. 11) can attain without breaking.

Take for wrought iron $\varphi = 14°$, $\gamma = 7.75$ and $\Omega = 10{,}000$ tons per square metre.

Substituting values in equation (2) we have that

$$A = \frac{4 \times 10000 \times \cos 14°}{7{,}75\,(1 - \text{sen}\ 14°)}$$

Therefore:

$$A = 6609 \text{ metres.}$$

For a block of granite with a apecific gravity of 2.72, an angle of friction of 33° and a breaking shearing atress of 4,000 tons per aquare metre; we should have that thet maximum height obtained would be 11686 metres.

In the case of a block of ice having a coefficient of friction equal to zero we shall have that the value of x in equation (1) becomes equal to A and therefore the ice will break at an angle of $\alpha = 45°$ which agrees with the result obtained in practice.

Note. The shearing stresses adopted in the above examples are only approximate values.

I have not succeeded in obtaining satisfactory ones.

Example. Find the maximum height that can be attained by a mass of dry earth of the shape shown in (fig. 11), having an angle of repose of 35°, and an ultimate shearing stress of 5 tons per equare metre. Take the specifiic gravity of earth as 1.5.

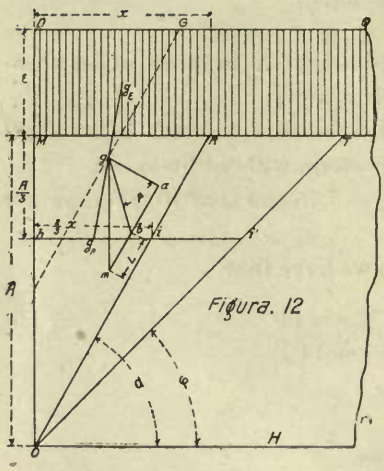
Substituting the known quantities in formula (2) we have that:

$$A = \frac{4 + 5 + \cos 35°}{1{,}5\ (1 - \text{sen}\ 35°)}$$

$$A = 25 \text{ metres.}$$

The value of the maximum unbalanced force can be found graphically by method given in paragraht **12**.

Problem N.° 3



14—*The surface of minimum resistance of a mass T M O H bounded by a horizontal and vertical plane such as M T and M O is a plane.*

Imagine a uniformly distributed load placed on top of the upper plane as shown in (fig. 12), and consider this load formed by small frictionless vertical prisms having no cohesion. Let *O R* be the position of the surface of minimum resistance.

The resultant weight of the prism and load acts at the centre of gravity g of the system, g_p and g_ε being the c. gs of the prism and load respectively.

The unbalanced force is given by:

$$L = W (\sin \alpha - \cos \alpha \tan \varphi)$$

where W is the sum of the two weights.

The shearing stress is:

$$\frac{L}{S} = \left(\varepsilon \gamma' + \frac{A \gamma}{2}\right)\left(\frac{A x - x^2 \tan \varphi}{A^2 + x^2}\right)$$

W_1 = weight of load resting along $M R = \varepsilon x \gamma'$.

Since for a maximum shearing stress we must equal the differential coefficient of

$$\frac{L}{S} \text{ ; to zero,}$$

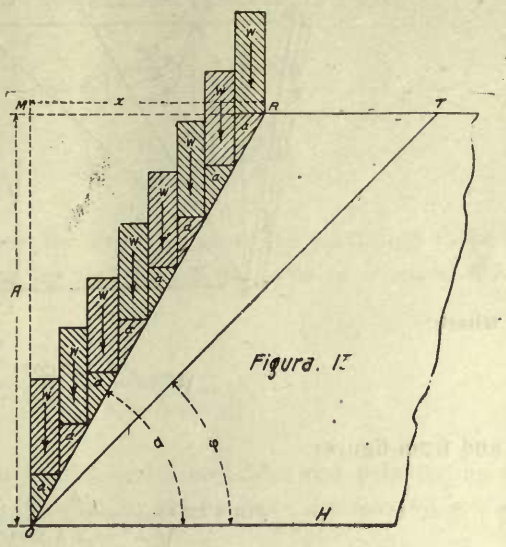
the factor,

$$\left(\varepsilon \gamma' + \frac{A \gamma}{2}\right)$$

has no influence whatever on the value of x, so we conclude that the position of the surface of rupture is independent of the weight W_1, resting on the mass.

15—Let us see now the conclutions that can be drawn from the above atatement.

In (fig. 13) let *O R* be the position of the surface of minimum resistance, and imagine the prism of condition taken away leaving a surface formed by small steps of width *a* as shown in figure. We have shown in paragraph **13** that the position of tbe planes of minimum resistance are parallel to each other, therefore for this case for the whole prism *R M O* and for each little prism of base *a* the surfaces of rupture are parallel. Let us place on each small step, concentrated equal loads *W* and imagine those loads be vertical prims. These loads have no influence on the position of plane of minimum resistance as shewn in paragraph **14**.



Now if we diminish the width *a* of the small prism forming the steps, it is clear that they will all break along the plane *O R*.

Therefore, for a mass of coherent substance the surface of rupture is a plane. Although to prove this proposition we have chosen a particular case the same can be said of any other mass having any other shape.

Problem N.° 4

16—*Determine the position of the plane of minimum resistance for a mass of a coherent body of the shape shown in (figure 14).*

Let *T M O N* be the mass of a coherent body bounded at the top by a horizontal plane *M T* and having a paramento inclined at an angle $\triangle$ with the vertical.

The general equation of the unbalanced force is:

$$L = W(\sin \alpha - \cos \alpha \tan \varphi)$$

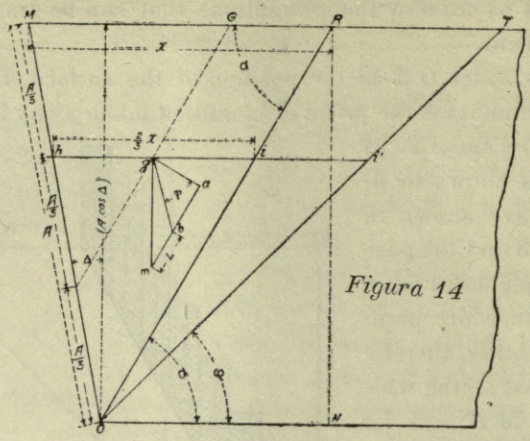


Figura 14

where;

$$W = \frac{A\ \gamma\ x\ cos\ \triangle}{2}$$

and from figure;

$$\sin \alpha = \frac{A \cos \triangle}{\sqrt{A^2 - 2\,A\,x \sin \triangle + x^2}}$$

$$\cos \alpha = \frac{x - A \sin \triangle}{\sqrt{A^2 - 2\,A\,x \sin \triangle + x^2}}$$

and;

$$S = \sqrt{A^2 - 2\,A\,x \sin \triangle + x^2}$$

Therefore;

$$(1)\ \frac{L}{S} = \frac{A\,\gamma \cos \triangle}{2 \cos \varphi}\left(\frac{A\,x \cos(\triangle - \varphi) - x^2 \operatorname{sen} \varphi}{A^2 - 2\,A\,x \sin \triangle + x^2}\right)$$

For a maximum $\dfrac{d\left(\frac{S}{L}\right)}{d\,x}$ must be zero

Now;

$$\frac{d\left(\frac{L}{S}\right)}{d\,x} = \frac{A^2\,\gamma \cos \triangle}{2 \cos \varphi} \times$$

$$\left(\frac{x^2(2 \sin \triangle \operatorname{sen} \varphi - \cos(\triangle - \varphi)) - 2\,A\,x \sin \varphi + A^2 \cos(\triangle - \varphi)}{(A^2 - 2\,A\,x \sin \triangle + x^2)^2}\right)$$

Therefore equating to zero we have:

$$x^2(2\sin\triangle\sin\varphi-\cos(\triangle-\varphi))-2Ax\sin\varphi+A^2\cos(\triangle-\varphi)=0$$

This is a quadratic equation whose roots are;

$$(2)\begin{cases} x_1 = A\left(\dfrac{\cos\triangle-\sin\varphi}{\cos(\triangle+\varphi)}\right) \\ x_2 - A\left(\dfrac{\cos\triangle+\sin\varphi}{\cos(\triangle+\varphi)}\right) \end{cases}$$

Out of these two values, the first one gives the maximum value for the shearing stress. Knowing value of x we have in triangle RNO that;

$$\tan\alpha = \frac{A\cos\triangle}{x-A\sin\triangle}$$

Therefore angle α can be obtained from tables and substituting value of x from equation (2) in equation (1) we have the maximum shearing stress for the plane of rupture.

Problem N°. 5

17—*Determine the position of the plane of minimum resistance for a mass of coherent body of the shape shown in figure 15).*

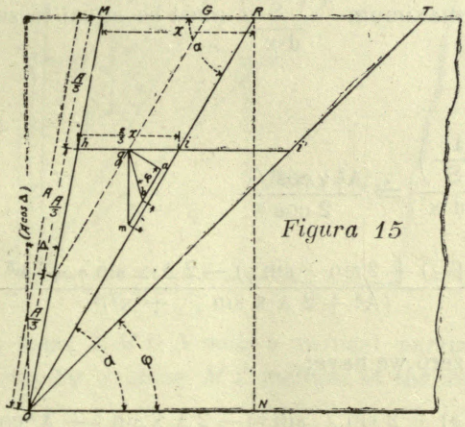


Figura 15

Consider now the same block as in previous problem but having the paramento $O\,M$ inclined to the vertical at a negative angle $\triangle$.

The unbalanced force is:

$$L = W\,(\sin\alpha - \cos\alpha \tan\varphi)$$

where;

$$W = \frac{A\,\gamma\,x\cos\triangle}{2}$$

In triangle $R\,N\,O$ we have that;

$$\sin\alpha = \frac{A\cos\triangle}{\sqrt{A^2 + 2\,A\,x\sin\triangle + x^2}}$$

$$\cos\alpha = \frac{x + A\sin\triangle}{\sqrt{A^2 + 2\,A\,x\sin\triangle + x^2}}$$

and;

$$S = \sqrt{A^2 + 2\,A\,x\sin\triangle + x^2}$$

Therefore;

$$(1) \qquad \left(\frac{L}{S}\right) = \frac{A\,\gamma\cos\triangle}{2\cos\varphi}\left(\frac{A\,x\cos(\triangle + \varphi) - x^2\sin\varphi}{A^2 + 2\,A\,x\sin\triangle + x^2}\right)$$

For a maximum $\dfrac{d\left(\frac{L}{S}\right)}{d\,x}$ must be aqual to zero.

Now;

$$\frac{d\left(\frac{L}{S}\right)}{d\,x} = \frac{A^2\,\gamma\cos\triangle}{2\cos\varphi}$$

$$\frac{-x^2(\cos(\wedge + \varphi) + 2\sin\triangle\sin\varphi) - 2\,A\,x\sin\varphi + A^2\cos(\triangle + \varphi)}{(A^2 + 2\,A\,x\sin\triangle + x^2)^2}$$

and equating to zero we have;

$$-x^2\,(\cos(\triangle + \varphi) + 2\sin\triangle\sin\varphi) - 2\,A\,x\sin\varphi + A^2\cos(\triangle + \varphi) = 0$$

This is a quadratic equation whose roots are:

$$x_1 = A\left(\frac{\cos\triangle - \sin\varphi}{\cos(\triangle - \varphi)}\right)$$

$$x_2 = -A\left(\frac{\cos\triangle + \sin\varphi}{\cos(\triangle - \varphi)}\right)$$

Out of these two values, the first one gives us the maximum shearing stress in equation (1)

Also having value of x in triangle R N O we have that:

$$\tan\alpha = \frac{A\cos\triangle}{x + A\sin\triangle}$$

Therefore angle α can be found from tables.

Problem No. 6

18—*Determine the position of the plane of minimum resistance for a mass of coherent body of the shape shown in (figure 16).*

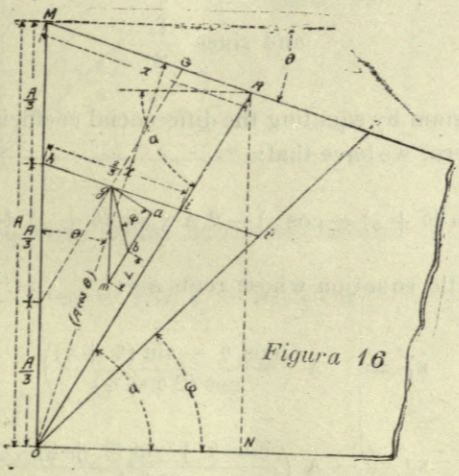


Figura 16

Consider a mass $T\,M\,O\,N$ with a vertical paramento M O and bounded at the top by a plane $M\,T$ inclined at the horizon at an angle θ.

In this case the weight of the prism cut out by the plane of minimum resistance is:

$$W = \frac{A \gamma x \cos \theta}{2}$$

an from figure in triangle $R N O$ we have that:

$$\sin \alpha = \frac{A - x \sin \theta}{\sqrt{A^2 - 2 A x \sin \theta + x^2}}$$

$$\cos \alpha = \frac{x \cos \theta}{\sqrt{A^2 - 2 A x \sin \theta + x^2}}$$

$$\text{also } S = \sqrt{A^2 - 2 A x \sin \theta + x^2}$$

Therefore substituting the above values in the general equation for the unbulanced force L and dividing by value of S we have that the shearing stress for plane α is:

$$(1) \qquad \frac{L}{S} = \frac{A \gamma \cos \theta}{2 \cos \varphi} \left(\frac{A x \cos \varphi - x^2 \sin (\theta + \varphi)}{A^2 - 2 A x \sin \theta + x^2} \right)$$

$$\text{and since } \frac{L}{S}$$

must be a maximum by equating the differencial coefficient of the above expression to zero, we have that:

$$x^2 (2 \sin \theta \sin (\theta + \varphi) - \cos \varphi) - 2 A x \sin (\theta + \varphi) + A^2 \cos \varphi = 0$$

This is a quadratic equation whose roots are:

$$x_1 = A \left(\frac{\cos \theta - \sin (\theta + \varphi)}{\cos (2 \theta + \varphi)} \right)$$

$$x_2 = - A \left(\frac{\cos \theta + \sin (\theta + \varphi)}{\cos (2 \theta + \varphi)} \right)$$

Out the these two values the first one gives us a maximum shearing stress in formula (1).

Since we found value of x in triangle $N\,O\,R$ we have that;

$$\tan \alpha = \frac{A - x \sin \theta}{x \cos \theta}$$

and therefore angle α can be obtained from tables.

Problem No. 7.

19—*Determine the position of the plane of minimum resistance for the mass of a coherent body having a vertical paramento and bounded at the top by a plane inclined at an angle* θ *above the horizon.* (fig. 17.)

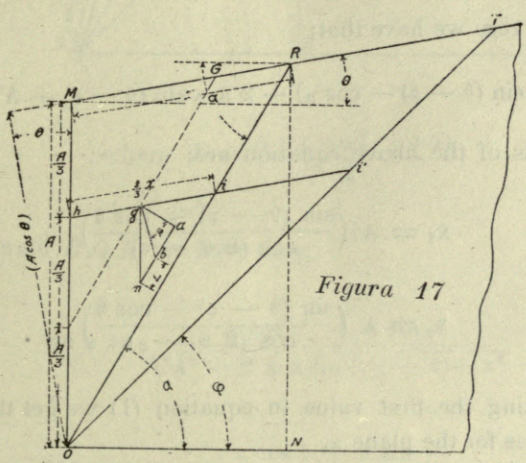


Figura 17

The general equation for the unbalancel force is;

$$L = W (\sin \alpha - \cos \alpha \tan \varphi)$$

where

$$W = \frac{A \gamma x \cos \theta}{2}$$

In triangle $R\,O\,N$ we have that;

$$\sin \alpha = \frac{A + x \sin \theta}{\sqrt{A^2 + 2\,A\,x \sin \theta + x^2}}$$

$$\cos \alpha = \frac{x \cos \theta}{\sqrt{A^2 + 2 A x \sin \theta + x^2}}$$

and: $$S = \sqrt{A^2 + 2 A x \sin \theta + x^2}$$

Therefore substituting the above values in the expressión $\frac{L}{S}$ we get that the shearing stress for the plane α is:

$$(1) \qquad \frac{L}{S} = \frac{A \gamma \cos \theta}{2 \cos \varphi} \left(\frac{A x \cos \varphi + x^2 \sin (\theta - \varphi)}{A^2 + 2 A x \sin \theta + x^2} \right)$$

and since $\frac{L}{S}$ must be a maximum; by equating the first differential coefficient to zero we have that:

$$x^2 (2 \sin \theta \sin (\theta - \varphi) - \cos \varphi) + 2 A x \sin (\theta - \varphi) + A^2 \cos \varphi = 0$$

The roots of the above equation are:

$$x_1 = A \left(\frac{\sin (\theta - \varphi) + \cos \theta}{\cos (2 \theta - \varphi)} \right)$$

$$x_2 = A \left(\frac{\sin (\theta - \varphi) - \cos \theta}{\cos (2 \theta - \varphi)} \right)$$

Substituting the first value in equation (1) we get the maximum shearing stress for the plane α.

Angle α is also obstained from the triangle $R\ O\ N$ where;

$$\tan \alpha = \frac{A + x \sin \theta}{x \cos \theta}$$

Problem No. 8

20—*Determine the plane of minimum resistance for a mass of a coherent body having the shape shewn in (figure* 18.)

The general equation of the unbalanced force is;

$$L = W (\sin \alpha - \cos \alpha \tan \varphi)$$

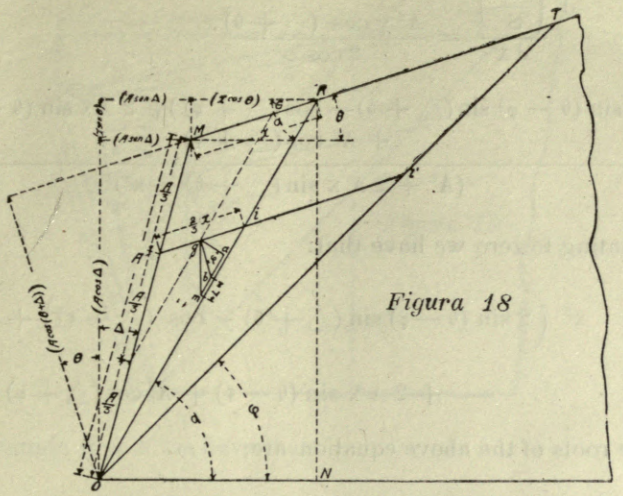


Figura 18

where $W = \frac{A\,\gamma\,x\cos(\triangle + \theta)}{2}$

In triangle $R\,O\,N$ we have that;

$$\sin\alpha = \frac{A\cos\triangle + x\sin\theta}{\sqrt{A^2 + 2\,A\,x\sin(\triangle + \theta) + x^2}}$$

$$\cos\alpha = \frac{x\cos\theta + A\sin\triangle}{\sqrt{A^2 + 2\,A\,x\sin(\triangle + \theta) + x^2}}$$

$$\text{and } S = \sqrt{A^2 + 2\,A\,x\sin(\triangle + \theta) + x^2}$$

Therefore substituting the above values in the general expression for the shearing force we have:

$$(1)\quad \frac{L}{S} = \frac{A\,\gamma\cos(\triangle + \theta)}{2\cos\varphi}\left(\frac{A\,x\cos(\triangle + \varphi) + x^2\sin(\theta - \varphi)}{A^2 + 2\,A\,x\sin(\triangle + \theta) + x^2}\right)$$

Differentiating for a maximum;

$$\frac{d\left(\frac{L}{S}\right)}{dx} = \frac{A^2 \gamma \cos(\triangle + \theta)}{2 \cos \varphi}$$

$$\left(\frac{x^2 (2 \sin(\theta - \varphi) \sin(\triangle + \theta) - \cos(\triangle + \varphi)) + 2 A x \sin(\theta - \varphi) + A^2 \cos(\triangle + \varphi)}{(A^2 + 2 A x \sin(\triangle + \theta) + x^2)^2} \right)$$

and equating to zero we have that:

$$x^2 \left(2 \sin(\theta - \varphi) \sin(\triangle + \theta) - \cos(\triangle + \varphi)\right) +$$

$$+ 2 A x \sin(\theta - \varphi) + A^2 \cos(\triangle + \varphi) = 0$$

The roots of the above equation are;

$$x_1 = A \left(\frac{\sin(\theta - \varphi) + \cos(\triangle + \theta)}{\cos(\triangle + 2\theta - \varphi)} \right)$$

$$x_2 = A \left(\frac{\sin(\theta - \varphi) - \cos(\triangle + \theta)}{\cos(\triangle + 2\theta - \varphi)} \right)$$

Out of these two values, the first one gives us a maximum for the shearing force in the plane of condition α. Therefore substituting its value in formula (1) and in the expression:

$$\tan \alpha = \frac{A \cos \triangle + x \sin \theta}{A \sin \triangle + x \cos \theta}$$

we get the maximum value for the shearing stress and the inclination of the plane of minimum resistance to the horizontal.

Problem N.° 9

21—*Determine the plane of minimum resistance for a mass of a coherent body having the shape shown in (figure* 19).

The general ecuation of the unbalanced force is:

$$L = W (\sin \alpha - \cos \alpha \tan \varphi)$$

where,

$$W = \frac{A \gamma x \cos(\triangle - \theta)}{2}$$

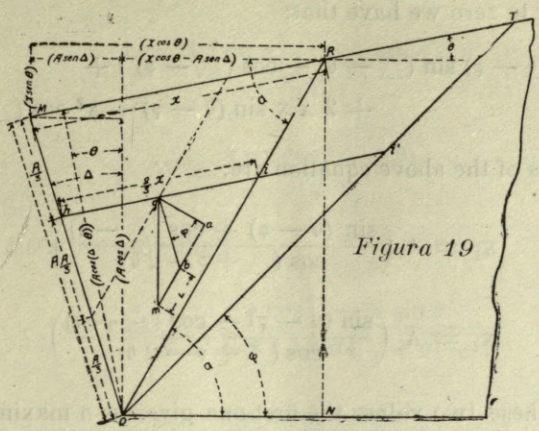


Figura 19

In triangle $R\ O\ N$ we have that:

$$\sin \alpha = \frac{A \cos \triangle + x \sin \theta}{\sqrt{A^2 - 2\ A\ x \sin (\triangle - \theta) + x^2}}$$

$$\cos \alpha = \frac{x \cos \theta - A \sin \triangle}{\sqrt{A^2 - 2\ A\ x \sin (\triangle - \theta) + x^2}}$$

and

$$S = \sqrt{A^2 - 2\ A\ x \sin (\triangle - \theta) + x^2}$$

Therefore substituting the above values in the general expression for the shearing force we have that:

$$(1) \qquad \frac{L}{S} = \frac{A\ \gamma \cos (\triangle - \theta)}{2 \cos \varphi} \left(\frac{A\ x \cos (\triangle - \varphi) + x^2 \sin (\theta - \varphi)}{A^2 - 2\ A\ x \sin (\triangle - \theta) + x^2} \right)$$

Differentiating for a maximum:

$$\frac{d \left(\frac{L}{S} \right)}{d\ x} = \frac{A^2\ \gamma \cos (\triangle - \theta)}{2 \cos \varphi}$$

$$\left(\frac{- x^2 (2 \sin (\theta - \varphi) \sin (\triangle - \theta) + \cos (\triangle - \varphi)}{A^2 - 2\ A\ x \sin (\triangle - \theta) + x^2} \right.$$

$$\left. \frac{+ 2\ A\ x \sin (\theta - \varphi) + A^2 \cos (\triangle - \theta)}{A^2 - 2\ A\ x \sin (\triangle - \theta) + x^2} \right)$$

and equating to zero we have that;

$$-x^2(2\sin(\theta-\varphi)\sin(\triangle-\theta)+\cos(\triangle-\varphi))+$$
$$+2Ax\sin(\theta-\varphi)+A^2\cos(\triangle-\varphi)=0$$

The roots of the above equation are;

$$x_1 = A\left(\frac{\sin(\theta-\varphi)+\cos(\triangle-\theta)}{\cos(\triangle+\varphi-2\theta)}\right)$$

$$x_2 = A\left(\frac{\sin(\theta-\varphi)-\cos(\triangle-\theta)}{\cos(\triangle+\varphi-2\theta)}\right)$$

Out of these two values the first one gives us a maximum for the shearing force in the plane of condition α. Therefore substituting this value in formula (1) and in the expression

$$\tan\alpha = \frac{A\cos\triangle + x\sin\theta}{x\cos\theta - A\sin\triangle}$$

we get the maximum value for the shearing stress and the inclination to the horizontal of the plane of minimum resistance.

Prpblem No. 10

22—*Determine the plane of minimum resistance for a mass of a coherent body of the shape shown in (figure 20).*

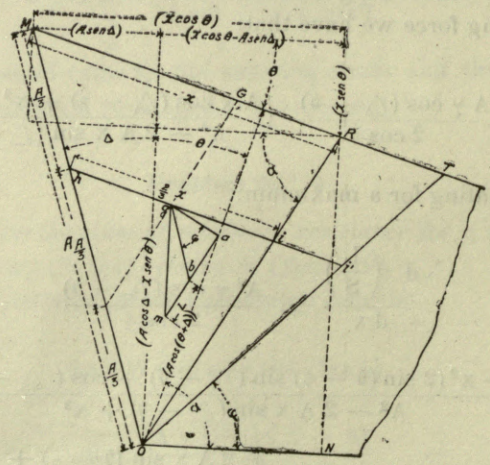


Figura 20

The general equation of the unbalanced force is:

$$L = W(\sin \alpha - \cos \alpha \tan \varphi)$$

where;

$$W = \frac{A \gamma x \cos(\triangle + \theta)}{2}$$

In triangle RON we have that:

$$\sin \alpha = \frac{A \cos \triangle - x \sin \theta}{\sqrt{A^2 - 2Ax \sin(\triangle + \theta) + x^2}}$$

$$\cos \alpha = \frac{x \cos \theta - A \sin \triangle}{\sqrt{A^2 - 2Ax \sin(\triangle + \theta) + x^2}}$$

and;

$$S = \sqrt{A^2 - 2Ax \sin(\triangle + \theta) + x^2}$$

Therefore substituting the above values in the general expression for the shearing force we have that;

$$(1) \quad \frac{L}{S} = \frac{A \gamma \cos(\triangle + \theta)}{2 \cos \varphi} \left(\frac{Ax \cos(\triangle - \varphi) - x^2 \sin(\theta + \varphi)}{A^2 - 2Ax \sin(\triangle + \theta) + x^2} \right)$$

Differentiating for a maximum we hawe:

$$\frac{d\left(\frac{L}{S}\right)}{dx} = \frac{A^2 \gamma \cos(\triangle + \theta)}{2 \cos \varphi}$$

$$\left(\frac{\begin{matrix} x^2(2 \sin(\theta + \varphi) \sin(\triangle + \theta) - \cos(\triangle - \varphi)) \\ -2Ax \sin(\theta + \varphi) + A^2 \cos(\triangle - \varphi) \end{matrix}}{(A^2 - 2Ax \sin(\triangle + \theta) + x^2)^2} - \right)$$

and equating to zero we get;

$$x^2(2 \sin(\theta + \varphi) \sin(\triangle + \theta) - \cos(\triangle - \varphi)) - 2Ax \sin(\theta + \varphi) +$$

$$+ A^2 \cos(\triangle - \varphi) = 0$$

whose roots are;

$$x_1 = A\left(\frac{\cos(\triangle + \theta) - \sin(\theta + \varphi)}{\cos(\triangle + \varphi + 2\theta)}\right)$$

$$x_2 = -A\left(\frac{\cos(\triangle + \theta) + \sin(\theta + \varphi)}{\cos(\triangle + \varphi + 2\theta)}\right)$$

Out of these two values, the first one gives us a maximum for the shearing force in the plane of condition α. Therefore substituting this value in formula (1) and in the expression

$$\tan \alpha = \frac{A \cos \triangle - x \sin \theta}{x \cos \theta - A \sin \triangle}$$

we obtain the maximum value for the shearing stress and the inclination to the horizontal of the plane of minimum resistance.

Problem N.° 11

23—*Determine the plane of minimum resistance for a mass of a coheren body having the shape shown in (figure 21).*

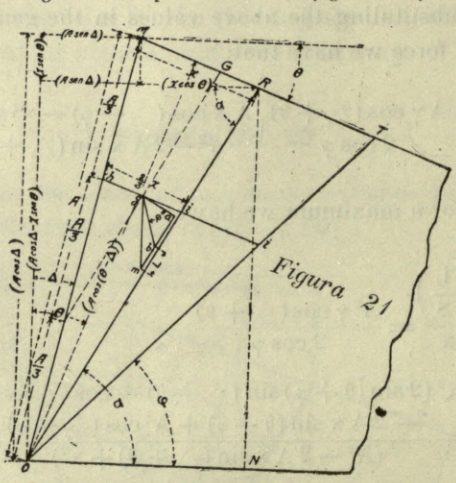


Figura 21

The general equation of the unbalanced force is;

$$L = W(\sin \alpha - \cos \alpha \tan \varphi)$$

where,

$$W = \frac{A \gamma x \cos(\theta - \triangle)}{2}$$

In triangle $R\ O\ N$ we have that;

$$\sin \alpha = \frac{A \cos \triangle - x \sin \theta}{\sqrt{A^2 - 2\,A\,x \sin(\theta - \triangle) + x^2}}$$

$$\cos \alpha = \frac{x \cos \theta + A \sin \triangle}{\sqrt{A^2 - 2\,A\,x \sin(\theta - \triangle) + x^2}}$$

and;

$$S = \sqrt{A^2 - 2\,A\,x \sin(\theta - \triangle) + x^2}$$

Therefore substituting the above values in the general expression for the shearing force in the plane of condition α we have;

$$(1)\quad \frac{L}{S} = \frac{A\,\gamma \cos(\theta - \triangle)}{2 \cos \varphi} \left(\frac{A\,x \cos(\triangle + \varphi) - x^2 \sin(\theta + \varphi)}{A^2 - 2\,A\,x \sin(\theta - \triangle) + x^2} \right)$$

Differentiating for a maximum we obtain;

$$\frac{d\left(\frac{L}{S}\right)}{dx} = \frac{A^2 \gamma \cos(\theta - \triangle)}{2 \cos \varphi}$$
$$\left(\frac{x^2 (2 \sin(\theta + \varphi) \sin(\theta - \triangle) - \cos(\triangle + \varphi)}{(A^2 - 2\,A\,x \sin(\theta - \triangle) + x^2)^2} \right)$$
$$\frac{- 2\,A\,x \sin(\theta + \varphi) + A^2 \cos(\triangle + \varphi)}{(A^2 - 2\,A\,x \sin(\theta - \triangle) + x^2)^2}$$

and equating to zero we have;

$$x^2 (2 \sin(\theta + \varphi) \sin(\theta - \triangle) - \cos(\triangle + \varphi)) - 2\,A\,x \sin(\theta + \varphi)$$
$$+ A^2 \cos(\triangle + \varphi) = 0$$

whose roots are;

$$x_1 = A \left(\frac{\cos(\theta - \triangle) - \sin(\theta + \varphi)}{\cos(\triangle - \varphi - 2\theta)} \right)$$

$$x_2 = -A \left(\frac{\cos(\theta - \triangle) + \sin(\theta + \varphi)}{\cos(\triangle - \varphi - 2\theta)} \right)$$

The first value of x gives us a maximum for the shearing force in the plane of condition α.

Therefore substituting this value in equation (1) and in the expression

$$\tan \alpha = \frac{A \cos \triangle - x \sin \theta}{x \cos \theta + A \sin \triangle}$$

we obtain the maximum value for the shearing stress and the inclination to the horizontal of the plane of minimum resistance.

Problem No. 12

24—*Determine the plane of minimum resistance for a mass of a coherent body bounded by a parabolic curve* (figure 22.)

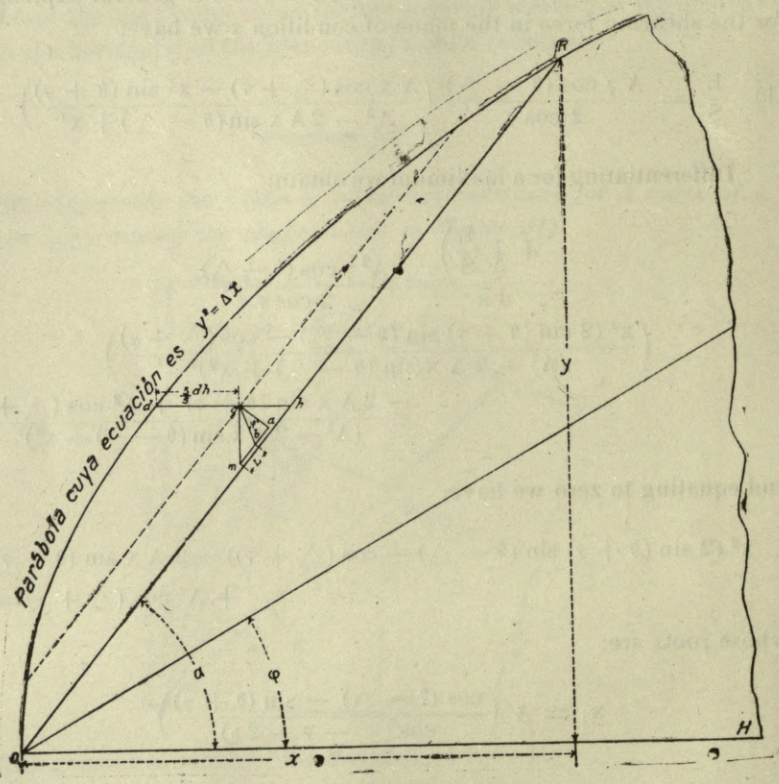


Figura 22

Let $y^2 = \Delta\ x$ be the equation of the parabolic surface bounding the mass *R M O H.*

Let (x, y) be the co-ordinates of the point R having O as origin.

The general equation of the unbalanced force is:

$$L = W\,(\sin\alpha - \cos\alpha \tan\varphi)$$

Now:

$$W = \left(\frac{2}{3}\,x\,y - \frac{x\,y}{2}\right)\gamma = \frac{x\,y\,\gamma}{6}$$

and:

$$y = x \tan\alpha$$

$$\sin\alpha = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\cos\alpha = \frac{x}{\sqrt{x^2 + y^2}}$$

and:

$$S = \sqrt{x^2 + y^2}$$

Substituting the above values in the general expression for the shearing stress we have that:

$$\frac{L}{S} = \frac{\gamma}{6}\left(\frac{\Delta\,x - x\sqrt{\Delta\,x}\tan\varphi}{x + \Delta}\right)$$

Differentiating for a maximum we obtain:

$$\frac{d\left(\frac{L}{S}\right)}{d\,x} = \frac{\Delta\,\gamma}{6}\left(\frac{2\,\Delta\,\sqrt{\Delta\,x} - x^2\tan\varphi - 3\,\Delta\,x\tan\varphi}{(x + \Delta)^2}\right)$$

and equating to zero we have that:

$$2\,\Delta\,\sqrt{\Delta\,x} - x^2\tan\varphi - 3\,\Delta\,x\tan\varphi = 0$$

Now let:

$$x = \omega + m$$

then:

$$\omega^3 - 3\ \omega\ \triangle^2 - \frac{34}{27}\triangle^3 - \frac{4\ \triangle^3}{\tan^2 \varphi} = 0$$

then:

$$x = \omega - 3\ \triangle$$

and:

$$\tan \alpha = \sqrt{\frac{\triangle}{x}}$$

CHAPTER V

Up to now, we have determined the plane of minimun resistance for masses bounded by surfaces that can be represented by equations. For bodies with irregular shapes or having some loads resting on them it is not possible to arrive at a mathematical formula for the determination of the plane of minimum resistance but a very approximate result can be obtained by trials, using graphic or analytic methods.

The following problems shall be treated in this way.

Problem N.° 13

25 *Determine the plane of minimum resistance for a mass of coherent body of an irregular shape as shown in (figure 23). with concentrated loads W and W_1 resting on it.*

26—Graphic Method. The plane of minimum resistance can be determined very approximately by trials as follows:

From point *O* draw a series of plane of conditions such as *Oi*, *Oj*, *O K*, etc., as shown in figure. Consider now the position of *Oi*.

Find the weight of the prism *a* and let *g m* to any scale represent this prism (fig. 23) (a). With *g m* as diametre draw a circle and from point *m* draw *m a* parallel to *O i* to meet circle in *a*. Join *a* and *g* and from *g* draw line *g b* at an angle φ (angle of repose) with *g a*: then *b m* gives us to scale the unbalanced force. Now the shearing force is obtained by dividing *b m* by *O i*.

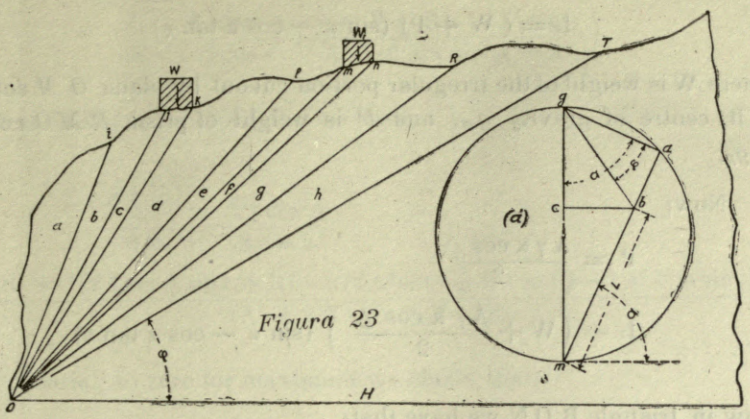


Figura 23

Do the same for prisms $(a + b)$, $(a + b + c + W)$ etc., etc. calculate the shearing stress in each case and the position that gives the maximun ratio of unbalanced force to area is that of the plane of minimun resistance required. It is quite clear that by taking the different positions very close to each other we may arrive as near as we like to the exact position of the plane of minimum resistance.

27—Analitic Method. This method is more laborious than the previous one, and can only be applied in certain cases.

Let $T\,M\,O$ (fig. 24) be a mass of coherent body having an irregu-

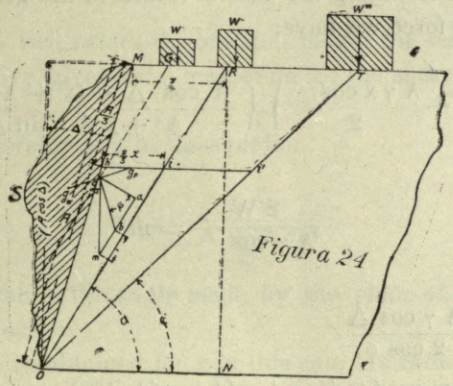


Figura 24

lar paramento as shown and bounded at the top by a horizontal plane *M. T.*

Join *O* and *M* and let angle $M\,O\,N$ be greater than the angle α made by the plane of minimun resistance with the horizon.

The general equation of the unbalanced force is;

$$L = (W + P)(\sin \alpha - \cos \alpha \tan \varphi)$$

where W is weight of the irregular portion cut out by plane $O\ M$ acting at its centre of gravity g_w, and P is weight of prism $R\ M\ O$ acting at g_p.

Now;

$$P = \frac{A \gamma x \cos \triangle}{2}$$

$$L = \left(W + \frac{A \gamma\ x \cos \Delta}{2}\right)(\sin \alpha - \cos \alpha \tan \varphi)$$

Now in triangle R O N we have that;

$$\sin \alpha = \frac{A \cos \Delta}{\sqrt{A^2 + 2 A x \sin \Delta + x^2}}$$

$$\cos \alpha = \frac{x + A \sin \Delta}{\sqrt{A^2 + 2 A x \sin \Delta + x^2}}$$

$$\text{also,} \qquad S = \sqrt{A^2 + 2\ A\ x \sin \Delta + x^2}$$

Therefore substituting the above values in the general expression for the shearing force we have;

$$\frac{L}{S} = \left(W + \frac{A \gamma x \cos \triangle}{2}\right)\left(\frac{A \cos \Delta - (x + A \sin \Delta) \tan \varphi}{A^2 + 2 A x \sin \Delta + x^2}\right)$$

Now let;

$$\frac{2 W}{A \gamma \cos \triangle} = m$$

Then;

$$\frac{L}{S} = \frac{A \gamma \cos \Delta}{2 \cos \varphi}$$

$$\left(\frac{A m \cos (\Delta + \varphi) + x (A \cos (\Delta + \varphi) - m \sin \varphi) - x^2 \sin \varphi}{A^2 + 2 A x \sin \Delta + x^2}\right)$$

Calling;

$$\cos (\Delta + \varphi) = F \text{ and } A \cos (\Delta + \varphi) - m \sin \varphi = R$$

we have that;

$$\frac{L}{S} = \frac{A\,\gamma\cos\Delta}{2\cos\varphi}\left(\frac{A\,m\,F + R\,x - x^2\sin\varphi}{A^2 + 2\,A\,x\sin\Delta + x^2}\right)$$

Differentiating and simplifying we have:

$$\frac{d\left(\frac{L}{S}\right)}{dx} = \frac{A\,\gamma\cos\Delta}{2\cos\varphi}$$

$$\left(\frac{A^2R - x^2(2\,A\sin\Delta\sin\varphi + R) - x(2\,A^2\sin\varphi + 2\,A\,m\,F) - 2\,A^2\,F\,m\sin\Delta}{(A^2 + 2\,A\,x\sin\Delta + x^2)^2}\right)$$

and equating to zero for maximum we obtain that:

$$A^2\,R - x^2\,(2\,A\sin\Delta\sin\varphi + R) - x\,(2\,A^2\sin\varphi + 2\,A\,m\,F)$$

$$- 2\,A^2\,F\,m\sin\Delta = 0$$

This is a quadratic equation whose roots are;

$$x_1 = A\left(\frac{A\sin\varphi + m\cos(\Delta + \varphi) - \cos\Delta\sqrt{A^2 + m^2 - 2\,A\,m\sin\triangle}}{m\sin\varphi - A\cos(\triangle - \varphi)}\right)$$

$$x_2 = A\left(\frac{A\sin\varphi + m\cos(\triangle + \varphi) + \cos\triangle\sqrt{A^2 + m^2 - 2\,A\,m\sin\triangle}}{m\sin\varphi - A\cos(\triangle - \varphi)}\right)$$

Out of these two values the one that makes the second differential coefficient negative gives us a maximum for the shearing force in the plane of condition α.

Substituting value of x in the equation:

$$\tan\alpha = \frac{A\cos\triangle}{x + A\sin\triangle}$$

we obtain from tables the angle made by the plane of minimum resistance with the horizon.

The equations obtained for x in this case are rather too long and the work would still be more complicated if we assume some concentrated loads W', W'', W''' etc., resting on the top of the mass. So we see that the graphic method applied previously is easier than the analytic one and has sufficient accuracy for practical purposes; so I advise the student to follow that one, for cases similar to those described in this chapter.

Problem No. 14

28 *Explain why a right prism of a stony material under compression in a testing machine, crushes, leaving two pyramides standing with the vertex opposed.*

Consider a right prism of any stony material such as granite, marble, etc., having a square section and tested for compression until it fails.

The relation between the height of the specimen and its width is given in Problem No. 2, paragraph **13** according to formula

$$x = A\left(\frac{1 - \sin \varphi}{\cos \varphi}\right)$$

for any given substance.

In (Figure 25) we denote by x the height of the block and by B the he distance from the vertical paramento to the point where the plane of rupture cuts the surface, i, e, the width of specimen, so the above formula becomes:

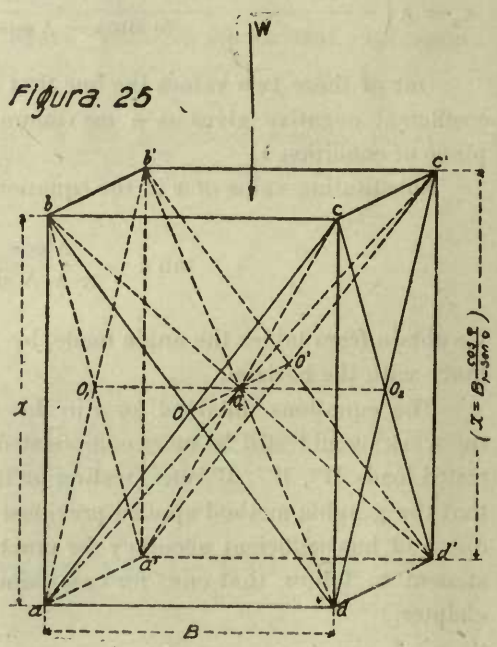
$$x = B\left(\frac{\cos \varphi}{1 - \sin \varphi}\right)$$

wich gives us the height of the block so that the planes of minimum resistance will be the diagonals of the faces of the specimen.

In prism $b\ c\ d'\ a'$ we have four planes of minimum resistance, as follows; $c\ c'\ a\ a'$, $b\ b'\ d\ d'$, $b'\ c'\ a\ d$ and $b\ c\ a'\ d'$.

Now suppose we place a uniformly distributed load on the top of the prism. This load does not alter the position of the planes of minimum resistance, as was shown in Problem No. 3, paragraph **14**. Let us increase this load

Figura. 25



little by little until the specimen fails. It is obvious that the block will fail simultaneously along the 4 planes of minimun resistance mentioned above and in so doing will break up in 14 solids having a common vertex at *Oc*, the centre of the prism.

The solids are;

Two equal pyramides having for bases the top and bottom faces of the block, *b b' c c'*, and *a a' d d'*.

Four equal solids *c d o o*$_2$ *Oc*, *b a o o*$_1$ *Oc*, *b' a' o*$_1$ *o' Oc*, *and c' d' o*$_2$ *o' Oc*.

Eight equal solids *b c o Oc*, *c c' o*$_2$ *Oc*, *b' c' o' Oc*, *b b' o*$_1$ *Oc*, *a d o Oc*, *d d' o*$_2$ *Oc*, *a' d' o' Oc and a a' o*$_1$ *Oc*.

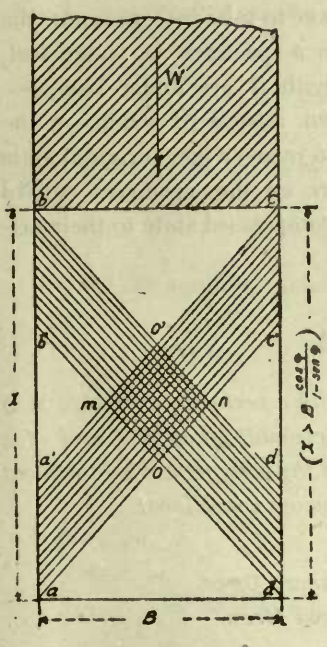
So theoretically, after the specimen has been taken out from the machine, we should find it split in 14 pieces as explained, but in practice we never get that for the reason that will be stated later on.

If the height x of the specimen is

$$x > B \left(\frac{\cos \varphi}{1 - \sin \varphi} \right)$$

it fails in a different way, (Fig. 26.)

Figura 26



The specimen will break leaving also 14 pieces but in a different disposition, 5 will have a common vertex at O' another 5 equal to the previous ones, will meet at *O* and the four remaining will be such as *m a' b'* one on each vertical corner of the prism and last, we shall have a zone such as the one shown shaded that will be crushed to powder,

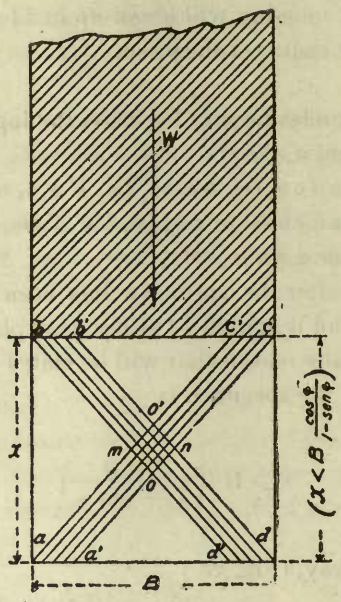
If the height, x of the specimen is

$$x < B \left(\frac{\cos \varphi}{1 - \sin \varphi} \right)$$

(fig. 27) it fails leaving also 14 pieces as in previous case, but the piramydes having their vertexes at O and O' will have smaller bases as shown in the figure.

In practice we never get the results as explained for several reasons. Materials are not perfectly homogeneous and conse-

Figura 27



quently the coefficient of friction and shearing stress are not constant for every direction and besides that, we have to take into account the causes due to the testing machine. When a specimen is under test, the parts of the machine in direct contact with it are under compression, and when the block fails this sudden release of strain on the plates of the machine causes the plates to recover their original state and in so doing they exert a second pressure on the specimen caused by the expansion of the plates from their compressed state to their original size.

Problem No. 15

29. *Given a mass of a coherent body with a vertical paramento* M O (*fig.* 28) *and bounded at the top by a horizontal plane* M T *and of a height* A *greater than that which will keep the body as a whole without breaking, determine the plane* x *for the following conditions:*

(a) *For a maximum unbalanced force.*

(b) *For a maximum horizontal umbalanced force.*

(c) *For a maximum vertical umbalanced force.*

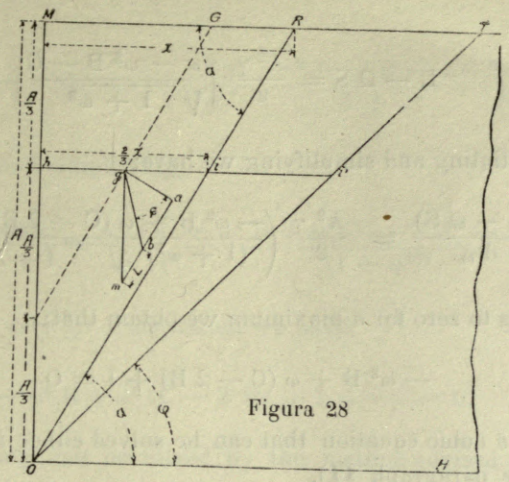


Figura 28

The resultant unbalanced force in this case would be the unbalanced force of the body considered as incoherent less the shearing force acting along plane of condition.

First Condition.—Maximum Unbalanced Force.

30—The general equation of the unbalanced force in Problem No. 1, paragraph **9**, at the end of page 13 is:

$$L = \frac{A\,\gamma}{2}\left(\frac{A\,x - x^2 \tan \varphi}{\sqrt{A^2 + x^2}}\right)$$

From the figure we have that:

$$S = \sqrt{A^2 + x^2}$$

Now the resultant unbalanced force is:

$$L - \Omega\,S = \frac{A\,\gamma}{2}\left(\frac{A\,x - x^2 \tan \varphi}{\sqrt{A^2 + x^2}}\right) - \Omega\,\sqrt{A^2 + x^2}$$

$$= \frac{A\,\gamma}{2}\left(\frac{A\,x - x^2\left(\tan \varphi + \frac{2\,\Omega}{A\,\gamma}\right) - \frac{2\,\Omega\,A^2}{A\,\gamma}}{\sqrt{A^2 + x^2}}\right)$$

Now let;

$$x = \omega\,A,\ \tan \varphi + \frac{2\,\Omega}{A\,\gamma} = B,\ \frac{2\,\Omega}{A\,\gamma} = C.$$

Then;

$$L - \Omega S = \frac{A^2 \gamma}{2} \left(\frac{\omega - \omega^2 B - C}{\sqrt{1 + \omega^2}} \right)$$

Differentiating and simplifying we have;

$$\frac{d (L - \Omega S)}{d \omega} = \frac{A^2 \gamma}{2} \left(\frac{- \omega^3 B + \omega (C - 2 B) + 1}{(1 + \omega^2) \sqrt{1 + \omega^2}} \right)$$

and equating to zero for a maximum we obtain that;

$$- \omega^3 B + \omega (C - 2 B) + 1 = 0$$

This is a cubic equation that can be solved either algebraically or by trials (see paragraph **11**).

By substituting value of x calculated in the expression:

$$\tan \alpha = \frac{A}{x}$$

angle α can be found from tables.

Second Condition. Maximum Horizontal Component.

31—The resultant unbalanced force is:

$$L - \Omega S = \frac{A \gamma}{2} \frac{\left(A x - x^2 \left(\tan \varphi + \frac{2 \Omega}{A \gamma} \right) - \frac{2 \Omega A^2}{A \gamma} \right)}{\sqrt{A^2 + x^2}}$$

The resultant horizontal component is:

$$(L - \Omega S) \cos \alpha = \frac{A \gamma}{2} \frac{\left(A x^2 - x^3 \left(\tan \varphi + \frac{2 \Omega}{A \gamma} \right) - \frac{2 \Omega A x}{A \gamma} \right)}{A^3 + x^2}$$

where:

$$\cos \alpha = \frac{x}{\sqrt{A^2 + x^2}}$$

Now, calling:

$$x = \omega A; \tan \varphi + \frac{2 \Omega}{A \gamma} = B; \text{ and } \frac{2 \Omega}{A \gamma} = C.$$

we have:

$$(L - \Omega S)\cos \alpha = \frac{A^2 \gamma}{2}\left(\frac{\omega^2 - \omega^3 B - \omega C}{1 + \omega^2}\right)$$

Differentiating and simplifying;

$$\frac{d\,((L - \Omega S)\cos \alpha)}{d\,\omega} = \frac{A^2 \gamma}{2}\left(\frac{-\omega^4 B + \omega^2 (C - 3B) + 2\omega - C}{(1 + \omega^2)^2}\right)$$

and equating to zero for a maximum we have:

$$-\omega^4 B + \omega^2 (C - 3B) + 2\omega - C = 0.$$

This equation can be solved by the method advised in Problem N.° 1, paragraph **11**.

Third Condition. Maximum Vertical Component.

32—The vertical component of the resultant unbalanced force is:

$$(L - \Omega S)\sin \alpha = \frac{A^2 \gamma}{2}\,\frac{\left(A x - x^2\left(\tan \varphi + \frac{2\Omega}{A\gamma}\right) - \frac{2\Omega A^2}{A\gamma}\right)}{A^2 + x^2}$$

where:

$$\sin \alpha = \frac{A}{\sqrt{A^2 + x^2}}$$

or substituting as in previous case, we have:

$$(L - \Omega S)\sin \alpha = \frac{A^2 \gamma}{2}\left(\frac{\omega - \omega^2 B - C}{1 + \omega^2}\right)$$

Differentiating and simplifying we obtain that:

$$\frac{d\,((L - \Omega S)\sin \alpha)}{d\,\omega} = \frac{A^2 \gamma}{2}\left(\frac{-\omega^2 + 2\omega (C - B) + 1}{1 + \omega^2)^2}\right)$$

and equating to zero for a maximum we get;

$$-\omega^2 + 2\omega (C - B) + 1 = 0$$

Putting values of C and B and solving we have that:

$$\omega_1 = \frac{1 - \sin \varphi}{\cos \varphi}$$

$$\omega^2 = \frac{1 + \sin \varphi}{\cos \varphi}$$

and;

$$\tan \alpha = \frac{A}{x}$$

and angle α can be obtained from tables.

So we conclude, that any plane cutting a coherent mass in aquilibrium receives lateral pressures, no matter the degree of coherence of the body provided that the mass has sufficient height A so that a prism will be displaced by the own weight and so doing acts on the plane considered.

CHAPTER VI

INCOHERENT BODIES

We have given in Chapter II, paragraph 4, the definition of an incoherent body. The desintegrated parts forming the body may vary in size from small to fine, according to the nature of the solid considered; but in any case they will have infinitesimal dimensions compared with the whole mass.

In practice, we never find incoherent bodies made up of infinitesimal particles, sand for instance, is formed by finite particles, but the size of the material making up the solid has no influence whatever on the way these particles are pressing against a plane.

Problem N.° 1

33—*Given a mass of an incoherent body with a vertical paramento M O (fig. 28) and bounded at the top by a horizontal plane M T, determine the plane α for the following conditions:*

(a) *For a maximum unbalanced force.*
(b) *For a maximum horizontal component.*
(c) *For a maximum vertical component.*

To solve this problem, we have to establish that the incoherent bodies exert pressures in a similar manner to the liquids with the only difference that for incoherent bodies we take into account the frictional force between particle and particle, while in the case of liquids this force is zero.

We shall also consider, as we did in **Problem N.° 3**, when dealing with the coherent bodies, that the surface of rupture is a plane. The reasoning in this case is similar to that one, except that it is not necessary in our case to place additional weights **(see fig. 13)** to break the small steps *a*, *a*, *a*.....

First Condition. Maximum Unbalanced Force

34—In an incoherent body, the tangential component *a m* of the weight of the prism R M O (fig. 28) is opposed only by the frictional force *b a* acting up the plane, so the resultant force acting down the incline is the unbalanced force.

Now in (**Problem N. 1**) of the coherent bodies we had that;

$$L = \frac{A\gamma}{2}\left(\frac{A\,x - x^2 \tan\varphi}{\sqrt{A^2 + x^2}}\right)$$

Differentiating and equating to zero for a maximum value of L we have;

$$\frac{d\,L}{d\,x} = \frac{A\gamma}{2}\left(\frac{A^3 - x^3\tan\varphi - 2\,A^2\,x\tan\varphi}{(A^2 + x^2)\sqrt{A^2 + x^2}}\right)$$

and:

$$A^3 - x^3\tan\varphi - 2\,A^2\,x\tan\varphi = 0$$

Solving this equation and taking value of x for a maximum value of L we have that;

$$x = A\left(\sqrt[3]{\frac{\cot\varphi}{2} + \sqrt{\frac{\cot^2\varphi}{4} + \frac{8}{27}}} + \sqrt[3]{\frac{\cot\varphi}{2} - \sqrt{\frac{\cot^2\varphi}{4} + \frac{8}{27}}}\right)$$

(see paragraph **9**, pag. 14).

Therefore substituting value of x in the expression

$$\tan \alpha = \frac{A}{x}$$

we have;

$$\tan \alpha = \frac{1}{\sqrt[3]{\frac{\cot \varphi}{2} + \sqrt{\frac{\cot^2 \varphi}{4} + \frac{8}{27}}} + \sqrt[3]{\frac{\cot \varphi}{2} - \sqrt{\frac{\cot^2 \varphi}{4} + \frac{8}{27}}}}$$

and angle α for the maximum unbalanced force can be obtained from tables.

Second Condition. Maximum Horizontal Component

35—The horizontal component of the unbalanced force is L Cos α.

$$L \cos \alpha = \frac{A \gamma}{2} \left(\frac{A x^2 - x^3 \tan \varphi}{A^2 + x^2} \right)$$

where:

$$\cos \alpha = \frac{x}{\sqrt{A^2 + x^2}}$$

Differentiating we have;

$$\frac{d (L \cos \alpha)}{d x} = \frac{A \gamma}{2} \left(\frac{2 A^3 x - 3 A^2 x^2 \tan \varphi - x^4 \tan \varphi}{(A^2 + x^2)^2} \right)$$

and equating to zero;

$$2 A^3 - 3 A^2 x \tan \varphi - x^3 \tan \varphi = 0$$

solving this equation, the value of x for a maximum is;

$$x = A \left(\sqrt[3]{\frac{\cos \varphi + 1}{\sin \varphi}} + \sqrt[3]{\frac{\cos \varphi - 1}{\sin \varphi}} \right)$$

(see paragraph **9** page 15)

and substituting this value in $\tan \alpha = \frac{A}{x}$ we have that:

$$\tan \alpha = \frac{1}{\sqrt[3]{\frac{\cos \varphi + 1}{\sin \varphi}} + \sqrt[3]{\frac{\cos \varphi - 1}{\sin \varphi}}}$$

and angle α for the maximum horizontal component can be found from tables.

Third Condition.—Maximum Vertical Component.

36—The vertical component of the unbalanced force is L sin α;

$$L \sin \alpha = \frac{A^2 \gamma}{2} \left(\frac{A x - x^2 \tan \varphi}{A^2 + x^2} \right)$$

where;

$$\sin \alpha = \frac{A}{\sqrt{A^2 + x^2}}$$

Differentiating we have;

$$\frac{d\ (L \sin \alpha)}{d x} = \frac{A^3 \gamma}{2} \left(\frac{A^2 - x^2 - 2 A x \tan \varphi}{(A^2 + x^2)^2} \right)$$

and equating to zero;

$$A^2 - x^2 - 2 A x \tan \varphi = 0$$

solving this equation the value of x for a maximum is;

$$x_1 = A \left(\frac{1 - \sin \varphi}{\cos \varphi} \right)$$

$$x_2 = -A \left(\frac{1 + \sin \varphi}{\cos \varphi} \right)$$

Therefore,

$$\tan \alpha = \frac{\cos \varphi}{1 - \sin \varphi}$$

aud angle α for maximum vertical component is obtained from tables.

Problem No. 2

37—*Given a mass of an incoherent body bounded at the top by a horizontal plane M T (fig. 29) and with an inclined paramento M O determine the plane α for the following condition.*

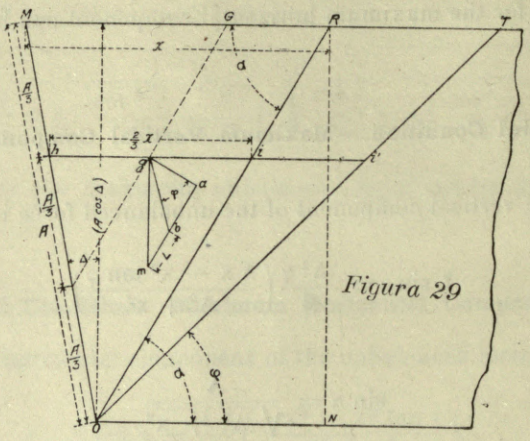


Figura 29

(a) *Maximum unbalanced force.*
(b) *Maximum horizontal component.*
(c) *Maximum vertical component.*

Firts Condition.—Maximum Unbalanced Force.

38—The general expression of the unbalanced force is:

$$L = W (\sin \alpha - \cos \alpha \tan \varphi)$$

From figure

$$W = \frac{A \gamma x \cos \triangle}{2}$$

$$\sin \alpha = \frac{A \cos \triangle}{\sqrt{A^2 - 2 A x \sin \triangle + x^2}}$$

and

$$\cos \alpha = \frac{x - A \sin \triangle}{\sqrt{A^2 - 2 A x \sin \triangle + x^2}}$$

Therefore substituting we have that;

$$L = \frac{A \gamma \cos \triangle}{2 \cos \varphi} \left(\frac{A x \cos (\triangle - \varphi) - x^2 \tan \varphi}{\sqrt{A^2 - 2 A x \sin \triangle + x^2}} \right)$$

Calling $x = \omega A$, $\cos(\triangle - \varphi) = B$, $\sin \varphi = C$, $\sin \triangle = H$. we have:

$$L = \frac{A^2 \gamma \cos \triangle}{2 \cos \varphi} \left(\frac{\omega B - \omega^2 C}{1 - 2\omega H + \omega^2} \right)$$

Differentiating we obtain;

$$\frac{dL}{d\omega} = \frac{A^2 \gamma \cos \triangle}{2 \cos \varphi} \left(\frac{-\omega^3 C + 3\omega^2 C H - \omega(B H + 2C) + B}{(1 - 2\omega H + \omega^2) \sqrt{1 - 2\omega H + \omega^2}} \right)$$

and equating to zero we get;

(1) $$-\omega^3 C + 3\omega^2 C H - \omega (B H + 2 C) + B = 0$$

This cubic equation has only one real value, (see paragraph **11** pag. 16), and that value gives us the maximum required.

To solve the above equation let $\omega = y + K$. Then substituting this value in (1) and replacing values on B, C and H we have that;

$$y^3 + \frac{y}{2} (4 \cos^2 \triangle + \sin (2 \triangle) \cot \varphi) - \frac{\cos^2 \triangle \cos (\triangle + \varphi)}{\sin \varphi} = 0$$

Calculating K we have that $K = \sin \triangle$.

And substituting value of x in

$$\tan \alpha = \frac{A \cos \triangle}{x - A \sin \triangle}$$

angle α can be calculated from tables.

Second Condition.—Maximum Horizontal Component.

39—The horizontal component of the unbalanced force is $L \cos \alpha$; Now;

$$L \cos \alpha = \frac{A \gamma \cos \triangle}{2 \cos \varphi}$$

$$\left(\frac{A x^2 (\cos (\triangle - \varphi) + \sin \triangle \sin \varphi) - x^3 \sin \varphi - A^2 x \cos (\triangle - \varphi) \sin \triangle}{A^2 - 2 A x \sin \triangle + x^2} \right)$$

Where:

$$\cos \alpha = \frac{x - A \sin \triangle}{\sqrt{A^2 - 2 A x \sin \triangle + x^2}}$$

calling;

$$\cos(\triangle - \varphi) + \sin \triangle \sin \varphi = B, \quad \sin \varphi = C, \quad \cos(\triangle - \varphi) \sin \triangle = F,$$

$$\sin \triangle = H, \qquad x = \omega A$$

we have that:

$$L \cos \alpha = \frac{A^2 \gamma \cos \triangle}{2 \cos \varphi} \left(\frac{\omega^2 B - \omega^3 C - \omega F}{1 - 2\omega H + \omega^2} \right)$$

Differentiating and simplifying;

$$\frac{d (L \cos \alpha)}{d \omega} = \frac{A^2 \gamma \cos \triangle}{2 \cos \alpha}$$

$$\left(\frac{-\omega^4 C + 4\omega^3 C H + \omega^2 (F - 2 B H - 3 C) + 2\omega B - F}{(1 - 2\omega H + \omega^2)^2} \right)$$

and equating to zero for a maximum we have that;

$$-\omega^4 C + 4\omega^3 C H + \omega^2 (F - 2 B H - 3 C) + 2\omega B - F = 0$$

This equation can be solved by trial (see paragraph **11**, pag. 16) and substituting value of x obtained in the expression.

$$\tan \alpha = \frac{A \cos \triangle}{x - A \sin \triangle}$$

α can be obtained from tables,

Third Condition. Maximum Vertical Component.

40—Thet vertical component of the unbalanced force is $L \sin \alpha$.

Now;

$$L = \frac{A \gamma \cos \triangle}{2 \cos \varphi} \left(\frac{A x \cos(\triangle - \varphi) - x^2 \sin \varphi}{\sqrt{A^2 - 2 A x \sin \triangle + x^2}} \right)$$

and;

$$\sin \alpha = \frac{A \cos \triangle}{\sqrt{A^2 - 2 A x \sin \triangle + x^2}}$$

Therefore;

$$L \sin \alpha = \frac{A^2 \gamma \cos^2 \triangle}{2 \cos \varphi} \left(\frac{A x \cos(\triangle - \varphi) - x^2 \sin \varphi}{A^2 - 2 A x \sin \triangle + x^2} \right)$$

calling;

$$x = \omega A, \qquad \cos(\triangle - \varphi) = B, \qquad \sin \varphi = C, \qquad \sin \triangle = H$$

we have that;

$$L \sin \alpha = \frac{A^2 \gamma \cos^2 \triangle}{2 \cos \varphi} \left(\frac{\omega B - \omega^2 C}{1 - 2 \omega H + \omega^2} \right)$$

Differentiating and simplifying we get;

$$\frac{d (L \sin \alpha)}{d \omega} = \frac{A^2 \gamma \cos^2 \triangle}{2 \cos \varphi} \left(\frac{\omega^2 (2 C H - B) - 2 \omega C + B}{(1 - 2 \omega H + \omega^2)^2} \right)$$

and equating to zero for a maximum we have;

$$\omega^2 (2 C H - B) - 2 \omega C + B = 0$$

Solving this equation we have:

$$\omega_1 = \frac{C}{2 C H - B} - \sqrt{\left(\frac{C}{2 C H - B} \right)^2 - B}$$

and

$$\omega_2 = \frac{C}{2 C H - B} + \sqrt{\left(\frac{C}{2 C H - B} \right)^2 - B}$$

Out of these two values, the one making the second differential coefficient of L negative gives us the maximum value of the unbalanced force L sin α and substituting the value obtained in the expression;

$$\tan \alpha = \frac{A \cos \triangle}{x - A \sin \triangle}$$

angle α can be obtained from tables.

Problem No. 3

41—*Given a mass of an incoherent body with the paramento making an angle* $\triangle$ *with the vertical (Fig. 30) and boundded at the top by a horizontal plane M T, determine the plane of condition* α *for the following conditions:*

(a) *Maximum unbalanced force.*
(b) *Maximum horizontal component.*
(c) *Maximum vertical component.*

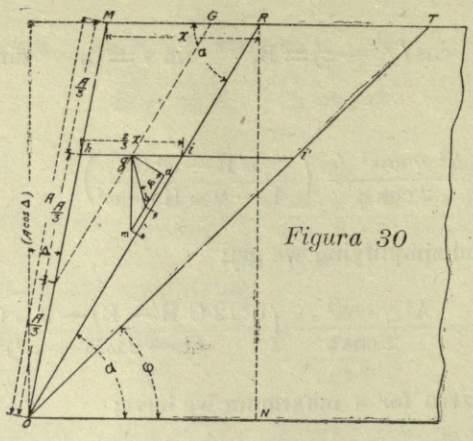


Figura 30

First Condition.—Maximum Unbalanced Force.

42—The general equation of the unbalanced force is:

$$L = W (\sin \alpha - \cos \alpha \tan \varphi)$$

From the figure

$$W = \frac{A \gamma x \cos \triangle}{2}$$

$$\sin \alpha = \frac{A \cos \Delta}{\sqrt{A^2 + 2 A x \sin \Delta + x^2}}$$

$$\cos \alpha = \frac{x + A \sin \Delta}{\sqrt{A^2 + 2 A x \sin \Delta + x^2}}$$

Therefore substituting the above values in the general equation we have;

$$L = \frac{A \gamma \cos \triangle}{2 \cos \varphi} \left(\frac{A x \cos (\triangle + \varphi) - x^2 \sin \varphi}{\sqrt{A^2 + 2 A x \sin \triangle + x^2}} \right)$$

Now making;

$$x = \omega A, \cos (\Delta + \varphi) = B, \sin \varphi = C, \sin \Delta = H$$

we get that;

$$L = \frac{A^2 \gamma \cos \triangle}{2 \cos \varphi} \left(\frac{\omega B - \omega^2 C}{\sqrt{1 + 2 \omega H + \omega^2}} \right)$$

Differentiating and simplifying we obtain;

$$\frac{d\,L}{d\,\omega} = \frac{A^2\ \gamma\ \cos\ \triangle}{2\ \cos\ \varphi}\left(\frac{-\omega^3\,C - 3\,\omega^2\,C\,H + \omega\,(B\,H - 2\,C) + B}{(1 + 2\,\omega\,H + \omega^2)\sqrt{1 + 2\,\omega\,H + \omega^2}}\right)$$

and equating to zero for maximum we have that;

$$-\omega^3\,C - 3\,\omega^2\,C\,H + \omega\,(B\,H - 2\,C) + B = 0$$

The above equation has only one real value, and this value gives us the maximum for the force L.

This equation can be solved by trials (see paragraph **11**—*pag.* 16) and to solve algebraically we get rid of the term in ω^2 by making

$$\omega = y + k$$

Therefore substituting in the equation of third degree the value of ω and replacing B, C and H for their values we obtain the following equation;

$$y^3 + \frac{y}{2}\,(4\cos^2\Delta - \sin\,(2\,\Delta)\,\cot\,\varphi) - \frac{\cos^2\,\triangle\,\cos\,(\triangle - \varphi)}{\sin\,\varphi} = 0$$

The value of $K = -\sin\ \triangle$, is obtained from calculations and substituting this value in equation $\omega = y + k$; ω is obtained and therefore x; and angle α from,

$$\tan\,\alpha = \frac{A\,\cos\,\triangle}{x + A\,\sin\,\triangle}$$

Second Condition. Maximum Horizontal Component.

43—The horizontal component of the unbalanced force is L cos α, therefore substituting their values we have;

$$L\,\cos\,\alpha = \frac{A\ \gamma\,\cos\ \triangle}{2\,\cos\,\varphi}$$

$$\left(\frac{A\,x^2(\cos\,(\triangle + \varphi) - \sin\,\triangle\,\sin\,\varphi) - x^3\,\sin\,\varphi + A^2\,x\,\cos\,(\triangle + \varphi)\,\sin\,\triangle)}{A^2 + 2\,A\,x\,\sin\,\triangle + x^2}\right)$$

Where;

$$\cos\,\alpha = \frac{x + A\,\sin\,\triangle}{\sqrt{A^2 + 2\,A\,x\,\sin\,\triangle + x^2}}$$

Now making:

$$x = \omega A,\ \sin \varphi = C,\ \sin \triangle = H,\ \cos (\triangle + \varphi) - \sin \triangle \sin \varphi = B$$

$$\cos (\triangle + \varphi) \sin \triangle = F.$$

we have;

$$L \cos \alpha = \frac{A^2 \gamma \cos \triangle}{2 \cos \varphi} \left(\frac{\omega^2 B - \omega^3 C + \omega F}{1 + 2 \omega H + \omega^2} \right)$$

Differentiating and simplifying;

$$\frac{d (L \cos \alpha)}{d \omega} = \frac{A^2 \gamma \cos \triangle}{2 \cos \varphi}$$

$$\left(\frac{\omega^4 C - 4 \omega^3 C H + \omega^2 (2 B H - F - 3 C) + 2 \omega B + F}{(1 + 2 \omega H + \omega^2)^2} \right)$$

and equating to zero for a maximum;

$$- \omega^4 C - 4 \omega^3 C H + \omega^2 (2 B H - F - 3 C) + 2 \omega B + F = 0$$

and solving the above equation as in previous problem (see paragraph **11** page 16) we find the value of x for a maximum horizontal component and consecuently angle α is obtained from equation;

$$\tan \alpha = \frac{A \cos \triangle}{x + A \sin \triangle}$$

Third Condition. Maximum Vertical Component.

44—The vertical component is L sin α therefore substituting values we have;

$$L \sin \alpha = \frac{A^2 \gamma \cos^2 \triangle}{2 \cos \varphi} \left(\frac{A x \cos (\triangle + \varphi) - x^2 \sin \varphi}{A^2 + 2 A x \sin \triangle + x^2} \right)$$

where;

$$\sin \alpha = \frac{A \cos \Delta}{\sqrt{A^2 + 2 A x \sin \triangle + x^2}}$$

Now making;

$$x = \omega A,\ \cos (\triangle + \varphi) = B,\ \sin \varphi = C,\ \sin \triangle = H$$

$$L \sin \alpha = \frac{A^2 \gamma \cos^2 \triangle}{2 \cos \varphi} \left(\frac{\omega B - \omega^2 C}{1 + 2 \omega H + \omega^2} \right)$$

Differentiating and simplifying:

$$\frac{d\ (L \sin \alpha)}{d\ \omega} = \frac{A^2 \gamma \cos^2 \triangle}{2 \cos \varphi} \left(\frac{-\omega^2 (B + 2 C H) - 2 \omega C + B}{(1 + 2 \omega H + \omega^2)^2} \right)$$

and equating to zero for a maximum we have:

$$-\omega^2 (B + 2 C H) - 2 \omega C + B = 0$$

solving the above equation we obtain for ω the following values:

$$\omega_1 = -\frac{C}{2 C H + B} + \sqrt{\left(\frac{C}{2 C H + B}\right)^2 + B}$$

$$\omega_2 = -\frac{C}{2 C H + B} - \sqrt{\left(\frac{C}{2 C H + B}\right)^2 + B}$$

Out of these two values, the one making the second differential coefficient negative gives us the maximum value for L sin α.

In this way x is calculated and consequently angle α is found from equation;

$$\tan \alpha = \frac{A \cos \triangle}{x + A \sin \triangle}$$

Problem No. 4

45—*Determine the plane of condition α for the three conditions enunciated in Problem No. 3 for an incoherent mas of the shape shewn in* (figure 31.)

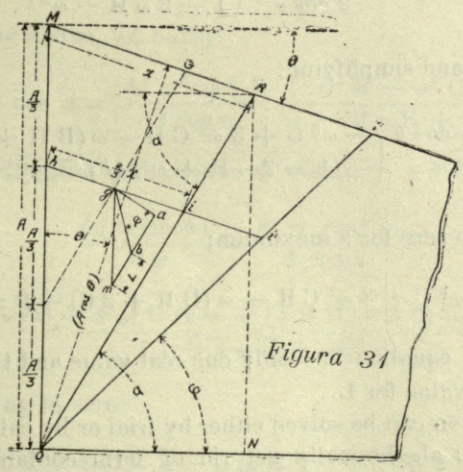


Figura 31

First Condition—Maximum Unbalanced Force

46—The general equation of the unbalanced force is;

$$L = W (\sin \alpha - \cos \alpha \tan \varphi)$$

From the figure;

$$W = \frac{A \gamma x \cos \theta}{2}$$

$$\sin \alpha = \frac{A - x \sin \theta}{\sqrt{A^2 - 2 A x \sin \theta + x^2}}$$

$$\cos \alpha = \frac{x \cos \theta}{\sqrt{A^2 - 2 A x \sin \theta + x^2}}$$

Therefore substituting the above values in the general equation of the unbalanced force we have that:

$$L = \frac{A \gamma \cos \theta}{2 \cos \varphi} \left(\frac{A x \cos \varphi - x^2 \sin (\theta + \varphi)}{\sqrt{A^2 - 2 A x \sin \theta + x^2}} \right)$$

Now making:

$$x = \omega A, \cos \varphi = B, \sin (\theta + \varphi) = C, \sin \theta = H$$

$$L = \frac{A^2 \gamma \cos \theta}{2 \cos \varphi} \left(\frac{\omega B - \omega^2 C}{1 - 2 \omega H + \omega^2} \right)$$

Differentiating and simplifying:

$$\frac{d L}{d \omega} = \frac{A^2 \gamma \cos \theta}{2 \cos \varphi} \left(\frac{- \omega^3 C + 3 \omega^2 C H - \omega (B H + 2 C) + B}{(1 - 2 \omega H + \omega^2) \sqrt{1 - 2 \omega H + \omega^2}} \right)$$

and equating to zero for a maximum:

$$- \omega^3 C + 3 \omega^2 C H - \omega (B H + 2 C) + B = 0$$

The above equation has only one real value and that one gives us the maximum value for L.

This equation can be solved either by trial or by calculations.

To solve it algebraically get rid of term containing ω^2 by ma-

king $\omega = y + k$ and substituting in the equation of third degree the value of ω and replacing B, C and H for their values we obtain the following equation;

$$y^3 + \frac{y}{2}(4\cos^2\theta + \sin(2\theta)\cot(\theta+\varphi)) - \frac{\cos^2\theta\cos(\varphi-2\theta)}{\sin(\theta+\varphi)} = 0$$

The value of $K = \sin\theta$, obtained by calculations is substituted in equation $\omega = y + k$, and therefore ω and consequently x, and α can be obtained; α from

$$\tan\alpha = \frac{A - x\sin\theta}{x\cos\theta}$$

Second Condition.—Maximum Horizontal Component.

47—The horizontal component of the unbalanced force is $L\cos\alpha$.

Now;

$$L\cos\alpha = \frac{A\gamma\cos^2\theta}{2\cos\varphi}\left(\frac{A x^2\cos\varphi - x^3\sin(\theta+\varphi)}{A^2 - 2Ax\sin\theta + x^2}\right)$$

where;

$$\cos\alpha = \frac{x\cos\theta}{\sqrt{A^2 - 2Ax\sin\theta + x^2}}$$

Now calling;

$$x = \omega A,\ \cos\varphi = B,\ \sin(\theta+\varphi) = C,\ \sin\theta = H$$

and substituting above, we have;

$$L\cos\alpha = \frac{A^2\gamma\cos^2\theta}{2\cos\varphi}\left(\frac{\omega^2 B - \omega^3 C}{1 - 2\omega H + \omega^2}\right)$$

Differentiating for a maximum we obtain;

$$\frac{d(L\cos\alpha)}{d\omega} = \frac{A^2\gamma\cos^2\theta}{2\cos\varphi}\times$$

$$\left(\frac{-\omega^4 C + 4\omega^3 C H - \omega^2(2BH + 3C) + 2\omega B}{(1 - 2\omega H + \omega^2)^2}\right)$$

and equating to zero;

$$-\omega^3 C + 4\omega^2 C H - \omega(2BH + 3C) + 2B = 0$$

The above equation can be solved as explained in (paragraph **11**, pag. 16) and value of α is obtained from the equation;

$$\tan \alpha = \frac{A - x \sin \theta}{x \cos \theta}$$

Third Condition.—Maximum Vertical Component.

48—The vertical component of the unbalanced force is, L sin α.

Now;

$$L \sin \alpha = \frac{A \gamma \cos \theta}{2 \cos \varphi} \times$$

$$\left(\frac{A^2 x \cos \varphi - A x^2 (\sin (\theta + \varphi) + \sin \theta \cos \varphi) + x^3 \sin (\theta + \varphi) \sin \theta}{A^2 - 2 A x \sin \theta + x^2} \right)$$

where;

$$\sin \alpha = \frac{A - x \sin \theta}{\sqrt{A^2 - 2 A x \sin \theta + x^2}}$$

Let us call;

$$x = \omega A, \cos \varphi = B, \sin (\theta + \varphi) + \sin \theta \cos \varphi = C, \sin (\theta + \varphi) \sin \theta = F$$
$$\sin \theta = H$$

Then substituting the above, we have;

$$L \sin \alpha = \frac{A^2 \gamma \cos \theta}{2 \cos \varphi} \left(\frac{\omega B - \omega^2 C + \omega^3 F}{1 - 2 \omega H + \omega^2} \right)$$

Differentiating and simplifying we get;

$$\frac{d(L \sin \alpha)}{d \omega} = \frac{A^2 \gamma \cos \theta}{2 \cos \varphi} \times$$

$$\left(\frac{\omega^4 F - 4 \omega^3 F H + \omega^2 (2 C H - B + 3 F) - 2 \omega C + B}{(1 - 2 \omega H + \omega^2)^2} \right)$$

and equating to zero for a maximum we obtain;

$$\omega^4 F - 4 \omega^3 F H + \omega^2 (2 C H - B + 3 F) - 2 \omega C + B = 0$$

The above equation can be solved as explained in (paragraph **11**, page 16) and value of angle α is obtained from the equation;

$$\tan \alpha = \frac{A - x \sin \theta}{x \cos \theta}$$

Problem N.° 5

49—*Given a mass of an incoherent body of the shape shown in* (figure 32) *determine the plane for the three conditions enunciated in Problem No. 3.*

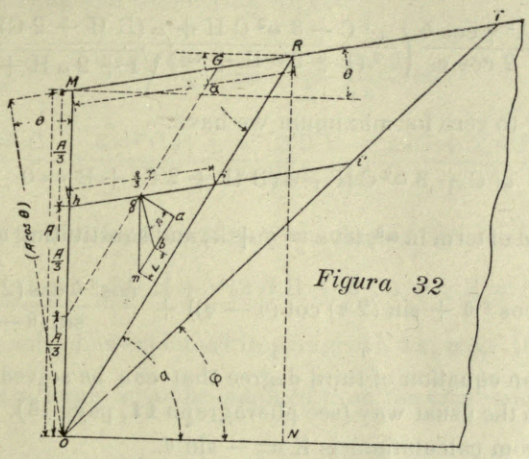


Figura 32

First Condition. Maximum Unbalanced Force.

50—The general equation of the unbalanced force is;

$$L = W (\sin \alpha - \cos \alpha \tan \varphi)$$

From the figure;

$$W = \frac{A \gamma x \cos \theta}{2}$$

$$\sin \alpha = \frac{A + x \sin \theta}{\sqrt{A^2 + 2 A x \sin \theta + x^2}}$$

$$\cos \alpha = \frac{x \cos \theta}{\sqrt{A^2 + 2 A x \sin \theta + x^2}}$$

Therefore substituting the above values in the general equation of the unbalanced force we have;

$$L = \frac{A \gamma \cos \theta}{2 \cos \varphi} \left(\frac{A x \cos \varphi + x^2 \sin (\theta - \varphi)}{\sqrt{A^2 + 2 A x \sin \theta + x^2}} \right)$$

Now let us call;

$$x = \omega A, \cos \varphi = B, \sin (\theta - \varphi) = C, \sin \theta = H.$$

and substituting above we have;

$$L = \frac{A^2 \gamma \cos \theta}{2 \cos \varphi} \left(\frac{\omega B + \omega^2 C}{\sqrt{1 + 2 \omega H + \omega^2}} \right)$$

Differentiating and simptifying we get;

$$\frac{d L}{d \omega} = \frac{A^2 \gamma \cos \theta}{2 \cos \varphi} \left(\frac{\omega^3 C + 3 \omega^2 C H + \omega (B H + 2 C) + B}{(1 + 2 \omega H + \omega^2) \sqrt{1 + 2 \omega H + \omega^2}} \right)$$

and equating to zero for maximum we have;

$$\omega^3 C + 3 \omega^2 C H + \omega (B H + 2 C) + B = 0$$

To get rid of term in ω^2, let $\omega = y + k$, and substituting above we get;

$$y^3 + \frac{y}{2} (4 \cos^2 \theta + \sin (2 \theta) \cot (\theta - \varphi)) + \frac{\cos^2 \theta \cos (2 \theta - \varphi)}{\sin (\theta - \varphi)} = 0$$

This is an equation of third degree that can be solved without any difficulting in the usual way (see pharagraph **11**, page 16). The value of *K* obtained from calculations is $K = - \sin \theta$.

Therefore knowing *x* angle α, can be obtained from equation;

$$\tan = \frac{x + x \sin \theta}{x \cos \theta}$$

Second Condition.—Maximum Horizontal Component.

51—The horizontal component of the unbalanced force is, $L \cos \alpha$ Now;

$$L = \frac{A \gamma \cos \theta}{2 \cos \varphi} \left(\frac{A x \cos \varphi + x^2 \sin (\theta - \varphi)}{\sqrt{A^2 + 2 A x \sin \theta + x^2}} \right)$$

$$\cos \alpha = \frac{x \cos \theta}{\sqrt{A^2 + 2 A x \sin \theta + x^2}}$$

Therefore,

$$L \cos \alpha = \frac{A \gamma \cos^2 \theta}{2 \cos \varphi} \left(\frac{A x^2 \cos \varphi + x^3 \sin (\theta - \varphi)}{A^2 + 2 A x \sin \theta + x^2} \right)$$

Now let;

$$x = \omega A,\ \cos\varphi = B,\ \sin(\theta - \varphi) = C,\ \sin\theta = H.$$

Therefore substituting above values in the equation of the force we have;

$$L\cos\alpha = \frac{A^2\gamma\cos^2\theta}{2\cos\varphi}\left(\frac{\omega^2 B + \omega^3 C}{1 + 2\omega H + \omega^2}\right)$$

Differentiating and simplifying we get;

$$\frac{d(L\cos\alpha)}{d\omega} = \frac{A^2\gamma\cos^2\theta}{2\cos\varphi} \times$$

$$\left(\frac{\omega^4 C + 4\omega^3 C H + \omega^2(2 B H + 3 C) + 2\omega B}{(1 + 2\omega H + \omega^2)^2}\right)$$

and equating to zero for a maximum we obtain;

$$\omega^3 C + 4\omega^2 C H + \omega(2 B H + 3 C) + 2 B = 0$$

which can be solved as explained in paragraph **11**, page 16.

Knowing x, angle α can be obtained from the expression;

$$\tan\alpha = \frac{A + x\sin\theta}{x\cos\theta}$$

Third Condition.—Maximum Vertical Component.

52—The vertical component of the unbalanced force is; $L\sin\alpha$

$$L\sin\alpha = \frac{A\gamma\cos\theta}{2\cos\varphi} \times$$

$$\left(\frac{A^2 x\cos\varphi + A x^2(\cos\varphi\sin\theta + \sin(\theta - \varphi)) + x^3\sin(\theta - \varphi)\sin\theta}{A^2 + 2 A x\sin\theta + x^2}\right)$$

where,

$$\sin\alpha = \frac{A + x\sin\theta}{\sqrt{A^2 + 2 A x\sin\theta + x^2}}$$

Now calling;

$$x = \omega A,\ \cos\varphi = B,\ \cos\varphi\sin\theta + \sin(\theta - \varphi) = C,\ \sin(\theta - \varphi)\sin\theta = F$$
$$\sin\theta = H$$

and substituting in the equation of the force we have;

$$L \sin \alpha = \frac{A^2 \gamma \cos \theta}{2 \cos \varphi} \left(\frac{\omega B + \omega^2 C + \omega^3 F}{1 + 2 \omega H + \omega 2} \right)$$

Differentiating and simplifying we get;

$$\frac{d (L \sin \alpha)}{d \omega} = \frac{A^2 \gamma \cos \theta}{2 \cos \varphi} \times$$

$$\left(\frac{\omega^4 F + 4 \omega^3 F H + \omega^2 (2 C H - B + 3 F) + 2 \omega C \times B}{(1 + 2 \omega H + \omega^2)^2} \right)$$

and equating to zero for a maximum we obtain;

$$\omega^4 F + 4 \omega^3 F H + \omega^2 (2 C H - B + 3 F) + 2 \omega C + B = 0$$

This is an equation of the 4th. degree which can be solved by trail as explained in paragraph, **11**, page 16.

Knowing x angle α can, be obtained from the equation;

$$\tan \alpha = \frac{A + x \sin \theta}{x \cos \theta}$$

Problem No. 6

53.—*Given a mass of an incoherent body of the shape shown in (figure 33) determine the plane α for the following conditions:*

(a) *Maximum unbalanced force.*
(b) *Maximum horizontal component.*
(c) *Maximum vertical component.*

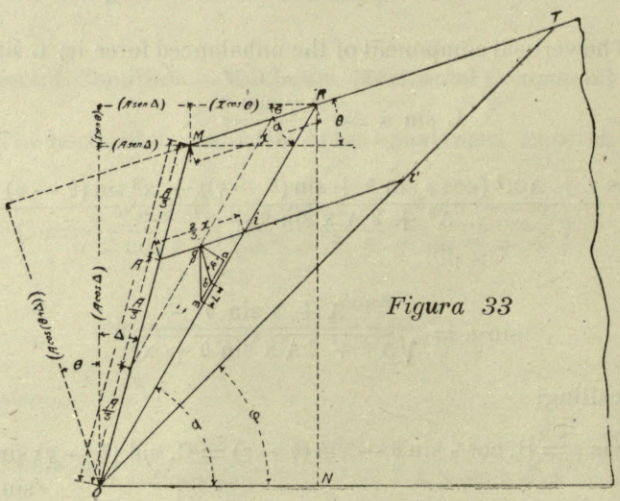


Figura 33

First Condition.—Maximum Unbalanced Force.

54—From the figure we have that;

$$W = \frac{A\,\gamma\,x\cos(\triangle + \theta)}{2}$$

$$\sin\alpha = \frac{A\cos\triangle + x\sin\theta}{\sqrt{A^2 + 2\,A\,x\sin(\triangle+\theta) + x^2}}$$

$$\cos\alpha = \frac{x\cos\theta + A\sin\triangle}{\sqrt{A^2 + 2\,A\,x\sin(\triangle+\theta) + x^2}}$$

Therefore substituting the above values in the general equation of the unbalanced force,

$$L = W(\sin\alpha - \cos\alpha\tan\varphi)$$

we have;

$$L = \frac{A\,\gamma\cos(\triangle+\theta)}{2\cos\varphi}\left(\frac{A\,x\cos(\triangle+\varphi) + x^2\sin(\theta-\varphi)}{\sqrt{A^2 + 2\,A\,x\sin(\triangle+\theta) + x^2}}\right)$$

Now calling;

$x = \omega\,A$, $\cos(\triangle+\theta) = B$, $\sin(\theta-\varphi) = C$, and $\sin(\triangle+\theta) = H$ and substituting in the general equation for L, we get;

$$L = \frac{A^2\,\gamma\cos(\triangle+\theta)}{2\cos\varphi}\left(\frac{\omega\,B + \omega^2\,C}{\sqrt{1 + 2\,\omega\,H + \omega^2}}\right)$$

Differentiating and simplifying we have;

$$\frac{dL}{d\omega} = \frac{A^2\,\gamma\cos(\triangle+\theta)}{2\cos\varphi}\left(\frac{\omega^3\,C + 3\,\omega^2\,C\,H + \omega(2\,C + B\,H) + B}{(1 + 2\,\omega\,H + \omega^2)\sqrt{1 + 2\,\omega\,H + \omega^2}}\right)$$

and equating to zero for a maximum we obtain;

$$\omega^3\,C + 3\,\omega^2\,C\,H + \omega(2\,C + B\,H) + B = 0$$

To solve above equation (see pharagraph **11**) let us make $\omega = y + k$ and in this way we get rid of the termin ω^2.

Substituting values of B. C. and H and simplying we get that;

$$y^3 + \frac{y}{2}(4\cos^2(\Delta+\theta) + \sin(2\theta+2\Delta)\cot(\theta-\varphi)) - \\ - \frac{\cos^2(\Delta+\theta)\cos(2\theta+\Delta-\varphi)}{\sin(\theta-\varphi)} = 0$$

The value of K obtained from calculations is; $K = -\sin(\Delta+\theta)$

In this way x can be obtained and angle α calculated from equation;

$$\tan\alpha = \frac{A\cos\Delta + x\sin\theta}{x\cos\theta + A\sin\Delta}$$

Second Condition. Maximum Horizontal Component.

55—The horizontal component of the unbalanced force is $L\cos\alpha$.

Now:

$$L = \frac{A\gamma\cos(\Delta+\theta)}{2\cos\varphi}\left(\frac{A x\cos(\Delta+\varphi) + x^2\sin(\theta-\varphi)}{\sqrt{A^2+2Ax\sin(\Delta+\theta)+x^2}}\right)$$

and;

$$\cos\alpha = \frac{x\cos\theta + A\sin\Delta}{\sqrt{A^2+2Ax\sin(\Delta+\theta)+x^2}}$$

Therefore;

$$L\cos\alpha = \frac{A\gamma\cos(\Delta+\theta)}{2\cos\varphi} \times$$

$$\left(\frac{A^2 x\cos(\Delta+\theta)\sin\Delta + Ax^2(\cos(\Delta+\varphi)\cos\theta + \sin(\theta-\varphi)\sin\Delta) + }{A^2+2Ax\sin(\Delta+\theta)+x^2}\right.$$

$$\left.\frac{+ x^3\sin(\theta-\varphi)\cos\theta}{A^2+2Ax\sin(\Delta+\theta)+x^2}\right)$$

Now, calling;

$x = \omega A$, $\cos(\Delta+\varphi)\sin\Delta = B$, $\cos(\Delta+\varphi)\cos\theta + \sin(\theta-\varphi)\sin\Delta = C$,
$\sin(\theta-\varphi)\cos\theta = F$, $\sin(\theta+\Delta) = H$

we have that;

$$L\cos\alpha = \frac{A^2\gamma\cos(\Delta+\theta)}{2\cos\varphi}\left(\frac{\omega B + \omega^2 C + \omega^3 F}{1+2\omega H+\omega^2}\right)$$

Differentiating and simplifying we have;

$$\frac{d\,(L\cos\alpha)}{d\,\omega} = \frac{A^2\,\gamma\cos(\triangle+\theta)}{2\cos\varphi} \times \left(\frac{\omega^4 F + 4\,\omega^3 F\,H + \omega^2\,(2\,C\,H - B + 3\,F) + 2\,\omega\,C + B}{(1 + 2\,\omega\,H + \omega^2)^2}\right)$$

and equating to zero for a maximum we obtain;

$$\omega^4 F + 4\,\omega^3 F\,H + \omega^2\,(2\,C\,H - B + 3\,F) + 2\,\omega\,C + B = 0$$

Solve this equation as explained in paragraph **11**.

Angle α can be obtained substituting value of x calculated, in equation;

$$\tan\alpha = \frac{A\cos\triangle + x\sin\theta}{x\cos\theta + A\sin\triangle}$$

Third Condition.—Maximum Vertical Component.

56—The vertical component of the unbalanced force is, $L\sin\alpha$.

Now;

$$L = \frac{A\,\gamma\cos(\triangle+\theta)}{A\cos\varphi}\left(\frac{A\,x\cos(\triangle+\varphi) + x^2\sin(\theta-\varphi)}{\sqrt{A^2 + 2\,A\,x\sin(\triangle+\theta) + x^2}}\right)$$

and;

$$\sin\alpha = \frac{A\cos\triangle + x\sin\theta}{\sqrt{A^2 + 2\,A\,x\sin(\triangle+\theta) + x^2}}$$

Therefore;

$$L\sin\alpha = \frac{A\,\gamma\cos(\triangle+\theta)}{2\cos\varphi} \times \left(\frac{\begin{array}{c}A^2\,x\cos(\triangle+\varphi)\cos\Delta + A\,x^2(\cos(\triangle+\varphi)\sin\theta + \sin(\theta-\varphi)\cos\triangle) \\ + x^3\sin(\theta-\varphi)\sin\theta\end{array}}{A^2 + 2\,A\,x\sin(\triangle+\theta) + x^2}\right)$$

Calling;

$$x = \omega\,A,\ \cos(\triangle+\varphi)\cos\Delta = B,$$

$$\cos(\triangle+\varphi)\sin\theta + \sin(\theta-\varphi)\cos\triangle = C,$$

$$\sin(\theta-\varphi)\sin\theta = F,\ \sin(\triangle+\theta) = H$$

we have that;

$$L \sin \alpha = \frac{A^2 \gamma \cos(\Delta + \theta)}{2 \cos \varphi} \left(\frac{\omega B + \omega^2 C + \omega^3 F}{1 + 2 \omega H + \omega^2} \right)$$

Differentiating and simplifying we get;

$$\frac{d (L \sin \alpha)}{d \omega} = \frac{A^2 \gamma \cos (\triangle + \theta)}{2 \cos \varphi} \times$$

$$\left(\frac{\omega^4 F + 4 \omega^3 F H + \omega^2 (2 C H - B + 3 F) + 2 \omega C + B}{(1 + 2 \omega H + \omega^2)^2} \right)$$

and equating to zero, for a maximum we obtain;

$$\omega^4 F + 4 \omega^3 F H + \omega^2 (2 C H - B + 3 F) + 2 \omega C + B = 0$$

This equation can be solved as explained in paragraph **11.**
Angle α can be obtained substituting value of x calculated, in equation;

$$\tan \alpha = \frac{A \cos \triangle + x \sin \theta}{x \cos \theta + A \sin \triangle}$$

Problem N.° 7

57. *Given a mass of an incoherent body of the shape shown in (figure 34), determine the plane for the following conditions;*

(a) *Maximum unbalance force.*
(b) *Maximum horizontal component.*
(c) *Maximum vertical component.*

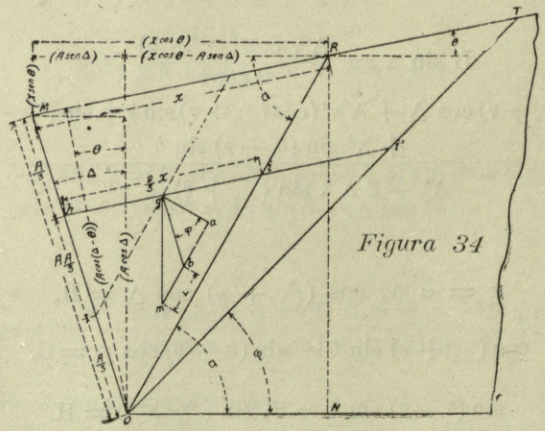


Figura 34

First Condition. Maximum Unbalanced Force

58. From the figure we have:

$$W = \frac{A\,\gamma\,x\cos(\Delta - \theta)}{2}$$

$$\sin\alpha = \frac{A\cos\Delta + x\sin\theta}{\sqrt{A^2 - 2\,A\,x\sin(\Delta - \theta) + x^2}}$$

$$\cos\alpha = \frac{x\cos\theta - A\sin\Delta}{\sqrt{A^2 - 2\,A\,x\sin(\Delta - \theta) + x^2}}$$

Terefore substituting the above values in the general equations of the unbalanced force;

$$L = W(\sin\alpha - \cos\alpha\tan\varphi)$$

we have;

$$L = \frac{A\,\gamma\cos(\Delta - \theta)}{2\cos\varphi}\left(\frac{A\,x\cos(\Delta - \varphi) + x^2\sin(\theta - \varphi)}{\sqrt{A^2 - 2\,A\,x\sin(\Delta - \theta) + x^2}}\right)$$

Now, calling;

$$x = \omega A,\ \cos(\Delta - \varphi) = B,\ \sin(\theta - \varphi) = C.,\ \sin(\Delta - \theta) = H$$

and substituting in the general equation for L, we get;

$$L = \frac{A^2\,\gamma\cos(\Delta - \theta)}{2\cos\varphi}\left(\frac{\omega\,B + \omega^2\,C}{1 - 2\,\omega\,H + \omega^2}\right)$$

Differentiating and simplifying;

$$\frac{dL}{d\omega} = \frac{A^2\,\gamma\cos(\Delta - \theta)}{2\cos\varphi}\left(\frac{\omega^3\,C - 3\,\omega^2\,C\,H + \omega(2\,C - B\,H) + B}{(1 - 2\,\omega\,H + \omega^2)\sqrt{1 - 2\,\omega\,H + \omega^2}}\right)$$

and equating to zero for a maximum we have;

$$\omega^3\,C - 3\,\omega^2\,C\,H + \omega(2\,C - B\,H) + B = 0$$

To get rid of ω^2, let $\omega = y + k$, and solve as explained in paragraph **11**.

Substituting values of B, C and H and simplifying, the above equation becomes;

$$y^3 + \frac{y}{2}\left(4\cos^2(\triangle - \theta) - \sin(2\triangle - 2\theta)\cot(\theta - \varphi)\right)$$

$$+ \frac{\cos^2(\triangle - \theta)\cos(\triangle + \varphi - 2\theta)}{\sin(\theta - \varphi)} = 0$$

The value of K obtained from calculations is $K = \sin(\triangle - \theta)$.
In this way x can be obtained and angle α calculated from equation.

$$\tan\alpha = \frac{A\cos\triangle + x\sin\theta}{x\cos\theta - A\sin\triangle}$$

Second Condition.—Maximum Horizontal Component.

59—The horizontal component of the unbalanced force is $L\cos\alpha$;

Now;

$$L = \frac{A\gamma\cos(\triangle - \theta)}{2\cos\varphi}\left(\frac{Ax\cos(\triangle - \varphi) + x^2\sin(\theta - \varphi)}{\sqrt{A^2 - 2Ax\sin(\triangle - \theta) + x^2}}\right)$$

and;

$$\cos\alpha = \frac{x\cos\theta - A\sin\triangle}{\sqrt{A^2 - 2Ax\sin(\triangle - \theta) + x^2}}$$

Therefore;

$$L\cos\alpha = \frac{A\gamma\cos(\triangle - \theta)}{2\cos\varphi} \times$$

$$\left(\frac{x^3\sin(\theta - \varphi)\cos\theta + Ax^2(\cos(\triangle - \varphi)\cos\theta - \sin(\theta - \varphi)\sin\triangle) - A^2x\cos(\triangle - \varphi)\sin\triangle}{A^2 - 2Ax\sin(\triangle - \theta) + x^2}\right)$$

Calling;

$$x = \omega A,\ \cos(\triangle - \varphi)\sin\triangle = B,\ \cos(\triangle - \varphi)\cos\theta - \sin(\theta - \varphi)\sin\triangle = C$$
$$\sin(\theta - \varphi)\cos\theta = F,\ \sin(\triangle - \theta) = H$$

we have that;

$$L\cos\alpha = \frac{A^2\gamma\cos(\triangle - \theta)}{2\cos\varphi}\left(\frac{-\omega B + \omega^2 C + \omega^3 F}{1 - 2\omega H + \omega^2}\right)$$

Differentiating and simplifying we get;

$$\frac{d\,(L\cos\alpha)}{d\,\omega} = \frac{A\,\gamma\cos(\triangle-\theta)}{2\cos\varphi}\times$$

$$\left(\frac{\omega^4 F - 4\,\omega^3 F H + \omega^2(B - 2\,C\,H + 3\,F) + 2\,\omega\,C - B}{(1 - 2\,\omega\,H + \omega^2)^2}\right)$$

and equating to zero for maximum we obtain;

$$\omega^4 F - 4\,\omega^3 F H + \omega^2(B - 2\,C\,H + 3\,F) + 2\,\omega\,C - B = 0$$

This equation can be solved as explained en paragraph **11**.

Angle α can be obtained substituting value of x calculated, in equation;

$$\tan\alpha = \frac{A\cos\triangle + x\sin\theta}{x\cos\theta - A\sin\triangle}$$

Third Condition. Maximum Vertical Component.

60—The vertical component of the unbalanced force is, $L\sin\alpha$;

Now;

$$L = \frac{A\,\gamma\cos(\triangle-\theta)}{2\cos\varphi}\left(\frac{A\,x\cos(\triangle-\varphi) + x^2\sin(\theta-\varphi)}{\sqrt{A^2 - 2\,A\,x\sin(\triangle-\theta+x^2}}\right)$$

and

$$\sin\alpha = \frac{A\cos\triangle + x\sin\theta}{\sqrt{A^2 - 2\,A\,x\,\mathrm{siu}\,(\triangle-\theta) + x^2}}$$

Therefore;

$$L\sin\alpha = \frac{A\,\gamma\cos(\triangle-\theta)}{2\cos\varphi}\times$$

$$\left(\frac{A^2\,x\cos(\triangle-\varphi)\cos\triangle + A\,x^2(\cos(\triangle-\varphi)\sin\theta + \sin(\theta-\varphi)\cos\triangle) + x^3\sin(\theta-\varphi)\sin\theta}{A^2 - 2\,A\,x\sin(\triangle-\theta) + x^2}\right)$$

Calling;

$$x = \omega\,A,\ \cos(\triangle-\varphi)\sin\theta + \sin(\theta-\varphi)\cos\triangle = C,\ \sin(\theta-\varphi)\sin\theta = F$$

$$\sin(\triangle-\theta) = H,\ \cos(\triangle-\varphi)\cos\triangle = B$$

we have that:

$$L \sin \alpha = \frac{A^2 \gamma \cos (\triangle - \theta)}{2 \cos \varphi} \left(\frac{\omega B + \omega^2 C + x^3 F}{1 - 2 \omega H + \omega^2} \right)$$

Differentiating and simplifying we get:

$$\frac{d(L \sin \alpha)}{d \omega} = \frac{A^2 \gamma \cos (\triangle - \theta)}{2 \cos \varphi} \times \left(\frac{\omega^4 F - 4 \omega^3 F H - \omega^2 (2 C H + B - 3 F) + 2 \omega C + B}{(1 - 2 \omega H + \omega^2)^2} \right)$$

and equatiating to zero for a maximum we obtain:

$$\omega^4 F - 4 \omega^3 F H - \omega^2 (2 C H + B - 3 F) + 2 \omega C + B = 0$$

This equation can be solved as explained in paragraph **11.**
Now, angle α can be obtained substituting value of x calculated, in the equation;

$$\tan \alpha = \frac{A \cos \triangle + x \sin \theta}{x \cos \theta - A \sin \triangle}$$

Problem N.° 8

61—*Given a mass of an incoherent body of the shape shown in (figure 35) determine the plane for the following conditions:*

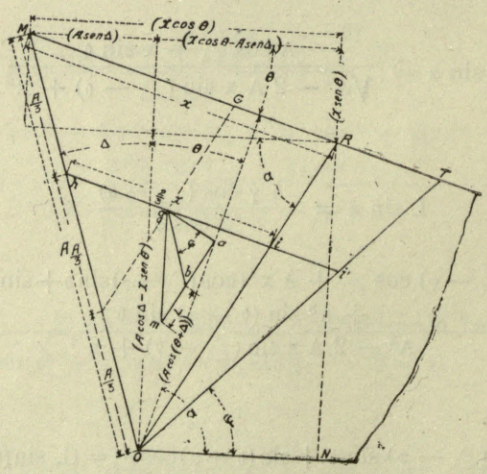


Figura 35

(a) *Maximum unbalanced force.*
(b) *Maximum horizontal component.*
(c) *Maximum vertical component.*

First Condition.—Maximum Unbalanced Force

62. From the figure we have:

$$W = \frac{A\,\gamma\,x\cos(\triangle + \theta)}{2}$$

$$\sin\alpha = \frac{A\cos\triangle - x\sin\theta}{\sqrt{A^2 - 2\,A\,x\sin(\triangle + \theta) + x^2}}$$

$$\cos\alpha = \frac{x\cos\theta - A\sin\triangle}{\sqrt{A^2 - 2\,A\,x\sin(\triangle + \theta) + x^2}}$$

Therefore substituting the above values in the general equation of the unbalanced force.

$$L = W(\sin\alpha - \cos\alpha\tan\varphi)$$

we have;

$$L = \frac{A\,\gamma\cos(\triangle + \theta)}{2\cos\varphi}\left(\frac{A\,x\cos(\triangle - \varphi) - x^2\sin(\theta + \varphi)}{\sqrt{A^2 - 2\,A\,x\sin(\theta + \triangle) + x^2}}\right)$$

Now calling;

$$x = \omega\,A,\ \cos(\triangle - \varphi) = B,\ \sin(\theta + \varphi) = C,\ \sin(\theta + \triangle) = H$$

we have;

$$L = \frac{A^2\,\gamma\cos(\triangle + \theta)}{2\cos\varphi}\left(\frac{\omega\,B - \omega^2\,C}{\sqrt{1 - 2\,\omega\,H + \omega^2}}\right)$$

Differentiating and simplifying we have;

$$\frac{dL}{d\omega} = \frac{A^2\,\gamma\cos(\triangle + \theta)}{2\cos\varphi}\left(\frac{-\omega^3\,C + 3\,\omega^2\,C\,H - \omega(2\,C + B\,H) + B}{(1 - 2\,\omega\,H + \omega^2)\sqrt{1 - 2\,\omega\,H + \omega^2}}\right)$$

and equating to zero for a maximum we obtain;

$$-\omega^3\,C + 3\,\omega^2\,C\,H - \omega(2\,C + B\,H) + B = 0$$

To solve this equation let us make $\omega = y + k$, and in this way we get rid of the term in ω^2.

Substituting values of B, C and H and simplifying we get;

$$y^3 + \frac{y}{2}(4\cos^2(\triangle + \theta) + \sin(2\theta + 2\triangle)\cot(\theta + \varphi))$$

$$- \frac{\cos^2(\triangle + \theta)\cos(\triangle + \varphi) + 2\theta)}{\sin(\theta + \varphi)} = 0$$

The value of K obtained from calculations is $K = \sin(\triangle + \theta)$

In this way x can be obtained and angle α calculated from equation;

$$\tan\alpha = \frac{A\cos\triangle - x\sin\theta}{x\cos\theta - A\sin\triangle}$$

Second Condition. Maximum Horizontal Component

63—The horizontal component of the unbalanced force is $L\cos\alpha$.

Now;

$$L = \frac{A\gamma\cos(\triangle + \theta)}{2\cos\varphi}\left(\frac{A x\cos(\triangle - \varphi) - x^2\sin(\theta + \varphi)}{\sqrt{A^2 - 2Ax\sin(\theta + \triangle) + x^2}}\right.$$

and;

$$\cos\alpha = \frac{x\cos\theta - A\sin\triangle}{\sqrt{A^2 - 2Ax\sin(\theta + \triangle) + x^2}}$$

Therefore;

$$L\cos\alpha = \frac{A\gamma\cos(\triangle + \theta)}{2\cos\varphi} \times$$

$$\left(\frac{\begin{array}{c}-A^2 x\cos(\triangle - \varphi)\sin\triangle + Ax^2(\cos(\triangle - \varphi)\cos\theta + \sin(\theta + \varphi)\sin\triangle) \\ - x^3\sin\theta + \varphi)\cos\theta\end{array}}{A^2 - 2Ax\sin(\theta + \triangle) + x^2}\right)$$

Now, calling;

$$x = \omega A, \cos(\triangle - \varphi)\sin\triangle = B, \cos(\triangle - \varphi)\cos\theta + \sin(\theta + \varphi)\sin\triangle = C$$

$$\sin(\theta + \varphi)\cos\theta = F, \sin(\theta + \triangle) = H$$

we have that;

$$L\cos\alpha = \frac{A^2\gamma\cos(\triangle + \theta)}{2\cos\varphi}\left(\frac{-\omega B + \omega^2 C - \omega^3 F}{1 - 2\omega H 4\omega^2}\right)$$

Differentiating and simplifying we get;

$$\frac{d\,(L\cos\alpha)}{d\,\omega} = \frac{A^2\,\gamma\cos(\triangle+\theta)}{2\cos\varphi}\times \left(\frac{-\omega^4 F+4\,\omega^3 F H+\omega^2(B-2CH-3F)+2\,\omega C-B}{(1-2\,\omega H+\omega^2)^2}\right)$$

and equating to zero for a maximum we obtain;

$$-\omega^4 F+4\,\omega^3 F H+\omega^2(B-2CH-3F)+2\,\omega C-B=0$$

Solve this equation as explained in paragraph **11.**
Angle α can be obtained substituting value of x calculated, in equation.

$$\tan\alpha = \frac{A\cos\triangle - x\sin\theta}{x\cos\theta - A\sin\triangle}$$

Third Condition.—Maximum Vertical Component.

64—The vertical component of the unbalanced force is, $L\sin\alpha$;

Now;

$$L = \frac{A\,\gamma\cos(\triangle+\theta)}{2\cos\varphi}\left(\frac{A\,x\cos(\triangle-\varphi)-x^2\sin(\theta+\varphi)}{\sqrt{A^2-2\,A\,x\sin(\triangle+\theta)+x^2}}\right)$$

and;

$$\sin\alpha = \frac{A\cos\triangle - x\sin\theta}{\sqrt{A^2-2\,A\,x\sin(\triangle+\theta)+x^2}}$$

Therefore;

$$L\sin\alpha = \frac{A\,\gamma\cos(\triangle+\theta)}{2\cos\varphi}\times \left(\frac{\begin{array}{c}A^2\,x\cos(\triangle-\varphi)\cos\triangle - A\,x^2\cos(\triangle-\varphi)\sin\theta \\ +A\,x^2\sin(\theta+\varphi)\cos\triangle + x^3\sin(\theta+\varphi)\sin\theta\end{array}}{A^2-2\,A\,x\sin(\Delta+\theta)+x^2}\right)$$

Now calling;

$$\cos(\triangle-\varphi)\cos\triangle = B,\ \cos(\triangle-\varphi)\sin\theta+\sin(\theta+\varphi)\cos\triangle = C,$$
$$\sin(\theta+\varphi)\sin\theta = F,\ \sin(\triangle+\theta) = H,\ x = \omega A.$$

we have that;

$$L\sin\alpha = \frac{A^2\,\gamma\cos(\triangle+\theta)}{2\cos\varphi}\left(\frac{\omega B-\omega^2 C+\omega^3 F}{1-2\,\omega H+\omega^2}\right)$$

Differenting and simplifying we get;

$$\frac{d(L \sin \alpha)}{d\ \omega} = \frac{A^2\ \gamma\ \cos\ (\Delta + \theta)}{2 \cos \varphi} \times$$

$$\left(\frac{\omega^4\ F - 4\ \omega^3\ F\ H + \omega^2\ (2\ C\ H - B + 3\ F) - 2\ \omega\ C + B}{(1 - 2\ \omega\ H + \omega^2)^{\ 2}}\right)$$

and equating to zero for a maximum we obtain;

$$\omega^4\ F - 4\ \omega^3\ F\ H + \omega^2\ (2\ C\ H - B + 3\ F) - 2\ \omega\ C + B = 0$$

This equation can be solved as explained in paragraph **11**, and value of x obtained, can be substituted in equation;

$$\tan \alpha = \frac{A\ \cos\ \triangle - x \sin \theta}{x\ \cos \theta - A \sin \triangle}$$

and angle α is obtained.

Problem No. 9

65—*Given a mass of an incoherent body of the shape shown in (figure 36) determine the plane α for the following conditions:*

(a) *Maximum unbalanced force.*
(b) *Maximum horizontal component.*
(c) *Maximum vertical component.*

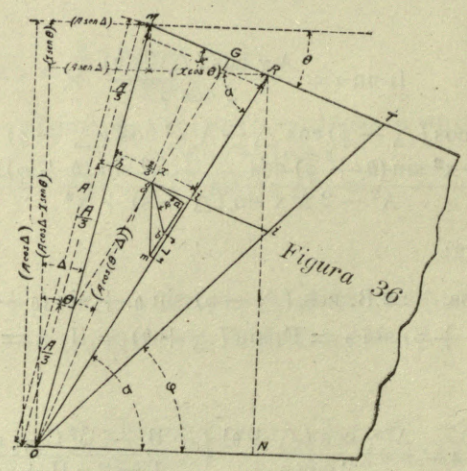


Figura 36

First Condition—Maximum Unbalanced Force

66—From the figure;

$$W = \frac{A\ \gamma\ x\ \cos(\theta - \triangle)}{2}$$

$$\sin\ \alpha = \frac{A\ \cos\ \triangle - x\ \sin\ \theta}{\sqrt{A^2 - 2\ A\ x\ \sin(\theta - \triangle) + x^2}}$$

$$\cos\ \alpha = \frac{x\ \cos\ \theta + A\ \sin\ \triangle}{\sqrt{A^2 - 2\ A\ x\ \sin(\theta - \triangle) + x^2}}$$

Therefore substituting the above values in the general equation of the unbalanced force;

$$L = W(\sin\ \alpha - \cos\ \alpha\ \tan\ \varphi)$$

we have;

$$L = \frac{A\ \gamma\ \cos(\theta - \Delta)}{2\ \cos\ \varphi}\left(\frac{A\ x\ \cos(\Delta + \varphi) - x^2\ \sin\ (\theta + \varphi)}{\sqrt{A^2 - 2\ A\ x\ \sin(\theta - \triangle) + x^2}}\right)$$

Now calling;

$$x = \omega\ A,\ \cos(\triangle + \varphi) = B,\ \sin(\theta + \varphi) = C,\ \sin(\theta - \triangle) = H$$

we have;

$$L = \frac{A^2\ \gamma\ \cos(\theta - \triangle}{2\ \cos\ \varphi}\left(\frac{\omega\ B - \omega^2\ C}{\sqrt{1 - 2\ \omega\ H + \omega^2}}\right)$$

Differentiating and simplifying we get;

$$\frac{d\ L}{d\ \omega} = \frac{A^2\ \gamma\ \cos\ (\theta - \triangle)}{2\ \cos\ \varphi}\left(\frac{-\ \omega^3\ C + 3\ \omega^2\ C\ H - \omega\ (2\ C + B\ H) + B}{(1 - 2\ \omega\ H + \omega^2)\ \sqrt{\quad 1 - 2\ \omega\ H + \omega^2}}\right)$$

and equating to zero for a maximum we obtain;

$$-\ \omega^3\ C + 3\ \omega^2\ C\ H - \omega\ (2\ C + B\ H) + B = 0$$

To solve this equation let us make $\omega = y + k$, and in this way we get rid of term in ω^2; substituting values of B, C and H and simplifying we get;

$$y^3 + \frac{y}{2}\left(4\cos^2(\theta - \triangle) + \sin(2\theta - 2\triangle)\cot(\theta + \varphi)\right) - \frac{\cos^2(\theta - \triangle)\cos(\triangle - \varphi - 2\theta)}{\sin(\theta + \varphi)} = 0$$

The value of K obtained from calculations is $K = \sin(\theta - \triangle)$.

In this way x can be calculated and angle α obtained from equation.

$$\tan\alpha = \frac{A\cos\triangle - x\sin\theta}{x\cos\theta + A\sin\triangle}$$

Second Condition.—Maximum Horizontal Component

67—The horizontal component of the unbalanced force is $L\cos\alpha$;

Now;

$$L = \frac{A\gamma\cos(\theta - \triangle)}{2\cos\varphi}\left(\frac{A x\cos(\triangle + \varphi) - x^2\sin(\theta + \varphi)}{\sqrt{A^2 - 2Ax\sin(\theta - \triangle) + x^2}}\right)$$

and;

$$\cos\alpha = \frac{x\cos\theta + A\sin\triangle}{\sqrt{A^2 - 2Ax\sin(\theta - \triangle) + x^2}}$$

Therefore;

$$L\cos\alpha = \frac{A\gamma\cos(\theta - \triangle)}{2\cos\varphi} \times \left(\frac{A^2 x\cos(\triangle + \varphi)\sin\triangle + Ax^2(\cos(\triangle + \varphi)\cos\theta - \sin(\theta + \varphi)\sin\triangle) - x^3\sin(\theta + \varphi)\cos\theta}{A^2 - 2Ax\sin(\theta - \triangle) + x^2}\right)$$

Now calling:

$$\cos(\triangle + \varphi)\sin\triangle = B,\ \cos(\triangle + \varphi)\cos\theta - \sin(\theta + \varphi)\sin\triangle = C,$$
$$\sin(\theta + \varphi)\cos\theta = F,\ \sin(\theta - \triangle) = H,\ x = \omega A.$$

we have that;

$$L\cos\alpha = \frac{A^2\gamma\cos(\theta - \triangle)}{2\cos\varphi}\left(\frac{\omega B + \omega^2 C - \omega^3 F}{1 - 2\omega H + \omega^2}\right)$$

Differentiating and simplifying we get;

$$\frac{d\,(L\cos\alpha)}{d\,\omega} = \frac{A^2\,\gamma\cos(\theta-\triangle)}{2\cos\alpha}\times$$
$$\left(\frac{-\omega^4 F+4\omega^3 F H-\omega^2\,(2\,C H+B+3 F)+2\,\omega\,C+B}{(1-2\,\omega H+\omega^2)^2}\right)$$

and equating to zero for a maximum we have;

$$-\omega^4 F+4\,\omega^3 F H-\omega^2\,(2\,C H+B+3 F)+2\,\omega\,C+B=0$$

Solve this equation as explained in paragraph **11,** and then angle α can be obtained by substituting the calculated value of x in the equation;

$$\tan\alpha = \frac{A\cos\Delta - x\sin\theta}{x\cos\theta + A\sin\Delta}$$

Third Condition.—Maximum Vertical Component.

68—Thet vertical component of the unbalanced force is $L\sin\alpha$.

Now;

$$L = \frac{A\,\gamma\cos(\theta-\triangle)}{2\cos\varphi}\left(\frac{A\,x\cos(\triangle+\varphi)-x^2\sin(\theta+\phi)}{\sqrt{A^2-2\,A\,x\sin(\theta-\triangle)+x^2}}\right)$$

and;

$$\sin\alpha = \frac{A\cos\triangle - x\sin\theta}{\sqrt{A^2-2\,A\,x\sin(\theta-\triangle)+x^2}}$$

Therefore;

$$L\sin\alpha = \frac{A\,\gamma\cos(\theta-\triangle)}{2\cos\varphi}$$
$$\left(\frac{A^2\,x\cos(\triangle+\varphi)\cos\triangle - A\,x^2(\cos(\triangle+\varphi)\sin\theta+\sin(\theta+\varphi)\cos\triangle)+ x^3\sin(\theta+\varphi)\sin\theta}{A^2-2\,A\,x\sin(\theta-\triangle)+x^2}\right)$$

Now calling;

$x=\omega\,A$, $\cos(\triangle+\varphi)\cos\triangle=B$, $\cos(\triangle+\varphi)\sin\theta+\sin(\theta+\varphi)\cos\triangle=C$,
$\sin(\theta+\varphi\sin\theta=F$, $\sin(\theta-\triangle)=H$.

we have that;

$$L \sin \alpha = \frac{A^2 \gamma \cos (\theta - \triangle)}{2 \cos \varphi} \left(\frac{\omega B - \omega^2 C + \omega^3 F}{1 - 2 \omega H + \omega^2} \right)$$

Differentiating and simplifying we get:

$$\frac{d (L \sin \alpha)}{d \omega} = \frac{A^2 \gamma \cos (\varphi - \triangle)}{2 \cos \varphi} \times$$

$$\left(\frac{\omega^4 F - 4 \omega^3 F H + \omega^2 (2 C H - B + 3 F) - 2 \omega C + B}{(1 - 2 \omega H + \omega^2)^2} \right)$$

and equating to zero for a maximum we have:

$$\omega^4 F - 4 \omega^3 F H + \omega^2 (2 C H - B + 3 F) - 2 \omega C + B = 0$$

This equation can be solved as explained in paragraph **11**, and angle α obtained by substituting calculated value of x in equation.

$$\tan \alpha = = \frac{A \cos \Delta - x \sin \theta}{x \cos \theta + A \sin \Delta}$$

Problem No. 10

69—*Given a mass of an incoherent substance of the shape shown in (figure* **37***) bounded at the top by a parabolic surface, determine the plane* α *for the following conditions:*

(a) *Maximum unbalanced force.*
(b) *Maximum horizontal component.*
(c) *Maximum vertical component.*

The equation of the parabola is;

$$y^2 = \triangle x$$

taking O as origen.

First Condition.—Maximum Unbalanced Force.

70—From the figure:

$$W = \left(\frac{2}{3} x y - \frac{y x}{2} \right) \gamma = \frac{x y \gamma}{6}$$

$$\sin \alpha = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\cos \alpha = \frac{x}{\sqrt{x^2 + y^2}}$$

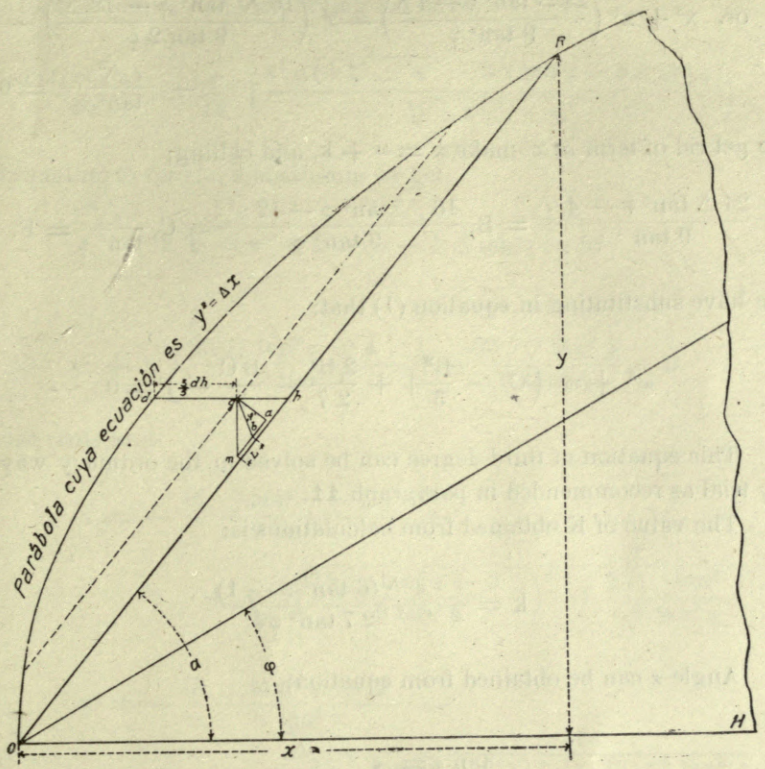


Figura 37

Therefore substituting the above values in the general equation of the unbalanced force;

$$L = W (\sin \alpha - \cos \alpha \tan \varphi)$$

we have;

$$L = \frac{\gamma}{6}\left(\frac{x^2 \Delta + x^2 \sqrt{\Delta x} \tan \varphi}{\sqrt{x^2 + \Delta x}}\right)$$

Differentiating and simplifying we get;

$$\frac{d L}{d x} = \frac{\gamma}{12} \times$$
$$\left(\frac{\Delta x^2 (\sqrt{4 \Delta x^3} + \sqrt{9 \Delta^3 x} - 3 x^2 \tan \varphi - 4 \Delta x \tan \varphi)}{\sqrt{\Delta x} (x^2 + \Delta x)^3}\right)$$

and equating to zero we have;

$$(1) \quad \sqrt{\Delta x} (2 x + 3 \Delta) - 3 x^2 \tan \varphi - 4 \Delta x \tan \varphi = 0$$

$$\text{or, } x^3 + x^2\left(\frac{24\,\triangle \tan^2\varphi - 4\,\triangle}{9\tan^2\varphi}\right) - x\left(\frac{16\,\triangle^2\tan^2\varphi - 12\,\triangle^2}{9\tan 2\varphi}\right) - \\ - \frac{\triangle^3}{\tan^2\varphi} = 0$$

To get rid of term in x^2 make $x = \omega + k$, and calling;

$$\frac{24\,\triangle\tan^2\varphi - 4\,\triangle}{9\tan^2\varphi} = B,\ \frac{16\,\triangle^2\tan^2\varphi - 12\,\triangle^2}{9\tan^2\varphi} = C,\ \frac{\triangle^3}{\tan^2\varphi} = F.$$

we have substituting in equation (1) that;

$$\omega^3 + \omega\left(C - \frac{B^2}{3}\right) + \frac{2B^3}{27} - \frac{BC}{3} - F = 0$$

This equation of third degree can be solved in the ordinary way or by trial as recommended in paragraph **11**.

The value of K obtained from calculations is;

$$k = -\frac{4\,\triangle(6\tan^2\varphi - 1)}{27\tan^2\varphi}$$

Angle α can be obtained from equation;

$$\tan\alpha = \sqrt{\frac{\triangle}{x}}$$

Second Condition.—Maximum Horizontal Component.

71—The horizontal component of the unbalanced force is $L\cos\alpha$.

Now;

$$L = \frac{\gamma}{6}\left(\frac{x^2\triangle + x^2\sqrt{\triangle x}\tan\varphi}{\sqrt{x^2 + \triangle x}}\right)$$

and;

$$\cos\alpha = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+\triangle x}}$$

Therefore;

$$L\cos\alpha = \frac{\gamma}{6}\left(\frac{\triangle x^2 - x^2\sqrt{\triangle x}\tan\varphi}{\triangle + x}\right)$$

Differentiating and simplifying we have;

$$\frac{d\,(L\cos\alpha)}{dx} = \frac{\gamma}{12}\left(\frac{x^2\triangle(2\sqrt{\triangle x} - 3x\tan\varphi - 5\triangle\tan\varphi)}{\sqrt{\triangle x(\triangle+x)^4}}\right)$$

and equating to zero for a maximum we get;

$$2\sqrt{\triangle x}\;\; 3x\tan\varphi - 5\triangle\tan\varphi = 0$$

or;

$$x^2 + x\left(\frac{30\,\triangle\tan^2\varphi - 4\,\triangle}{9\tan^2\varphi}\right) + \frac{25\,\triangle^2\tan^2\varphi}{9\tan^2\varphi} = 0$$

whose roots are;

$$x_1 = -\frac{15\,\triangle\tan^2\varphi - 2\,\triangle}{9\tan^2\varphi} + \\ + \sqrt{\left(\frac{15\,\triangle\tan^2\varphi - 2\,\triangle}{9\tan^2\varphi}\right)^2 - \frac{25\,\triangle^2\tan^2\varphi}{9\tan^2\varphi}}$$

$$x^2 = -\frac{15\,\triangle\tan^2\varphi - 2\,\triangle}{9\tan^2\varphi} - \\ - \sqrt{\left(\frac{15\,\triangle\tan^2\varphi - 2\,\triangle}{9\tan^2\varphi}\right)^2 - \frac{25\,\triangle^2\tan^2\varphi}{9\tan^2\varphi}}$$

Out of these two values the one making negative the second differential coefficient gives us the maximum value for the horizontal component.

Substituting value of x calculated, in the equation $\tan\alpha = \sqrt{\frac{\triangle}{x}}$ angle α is obtained.

Third Condition.—Maximum Vertical Component.

72—The vertical component of the unbalanced force, L sin α.

Now;

$$L = \frac{\gamma}{6}\left(\frac{\triangle x^2 + x^2\sqrt{\triangle x}\,\tan\varphi}{\sqrt{x^2+\triangle x}}\right)$$

and;

$$\sin \alpha = \frac{y}{\sqrt{x^2 + y^2}} = \frac{\sqrt{\triangle x}}{\sqrt{x^2 + \triangle x}}$$

Therefore;

$$L \sin \alpha = \frac{\gamma}{6} \left(\frac{\sqrt{\triangle^3 x^3} - \triangle x^2 \tan \varphi}{\Delta + x} \right)$$

Differentiating and simplifying we get:

$$\frac{d(L \sin \alpha)}{d x} = \frac{\gamma}{12} \times$$

$$\left(\frac{\Delta^2 x(\triangle x + 3 \triangle^2 - 2 \sqrt{\Delta x}\ x \tan \varphi - 4 \sqrt{\Delta^3 x} \tan \varphi)}{\sqrt{\triangle x (\triangle + x)^4}} \right)$$

and equating to zero for a maximum we have;

$$\triangle x + 3 \triangle^2 - 2 \sqrt{\triangle x}\ x \tan \varphi - 4 \sqrt{\triangle^3 x}\ \tan \varphi = 0$$

or;

$$x^3 + x^2 \left(\frac{16 \tan^2 \varphi \triangle - \triangle}{4 \tan^2 \varphi} \right) +$$

$$+ x \left(\frac{16 \tan^2 \varphi \triangle^2 - \triangle^2}{4 \tan^2 \varphi} \right) - \frac{9 \triangle^3}{4 \tan^2 \varphi} = 0$$

which can be solved as recomended in paragrahh **11**.

Knowing x, angle α can be obtained from equation $\tan \alpha = \sqrt{\frac{\triangle}{x}}$

We have studied so far, the maximum pressures on prisms bounded by surfaces that can be represented by algebraical equations, we are going to deal now with prisms of irregular shapes and prisms having concentrated loads resting on them.

Problem N.° 11

73—*Given a mass of an incoherent substance of the shape shown in (figure 38) bounded by an irregular surface and having concentrated loads as shown, determine the plane for the following conditions:*

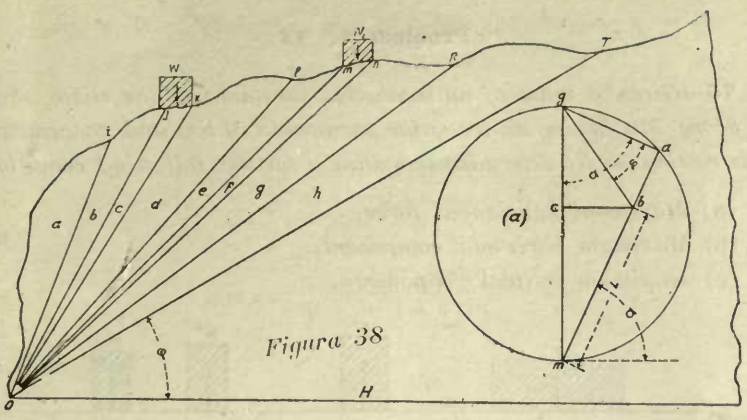


Figura 38

(a) *Maximum unbalanced force.*
(b) *Maximum horizontal component.*
(c) *Maximum vertical component.*

74—We suppose the irregular shape to be generated by a line perpendicular to the plane of the paper.

Since the prism is irregular, we cannot represent algebraically the weight and cannot express the forces in terms of the dimensions, and coefficient of friction, so we must proceed by trial and graphically.

Consider first the position of plane x along $O\,i$. Calculate the weight of the portion a separated by $O\,i$, and to any scale draw line $g\,m$ (fig. 38) (a) to represent this weight. From point m draw line $m\,a$ parallel to $O\,i$, till it meets circle drawn with $g\,m$ as diameter. Join a with g, and from g draw line $g\,b$ making with $g\,a$ angle of repose φ, then $b\,m$ gives us to scale the unbalanced force L.

From b draw horizontal line $b\,c$, then we have that;

$b\,c$ = horizontal component.

and, $c\,m$ = vertical component.

Do the same for prisms $(a+b)$, $(a+b+c+W)$ and so on, until we obtain the positions of x that give us to scale the maximum values of the unbalanced force; horizontal component, and vertical component.

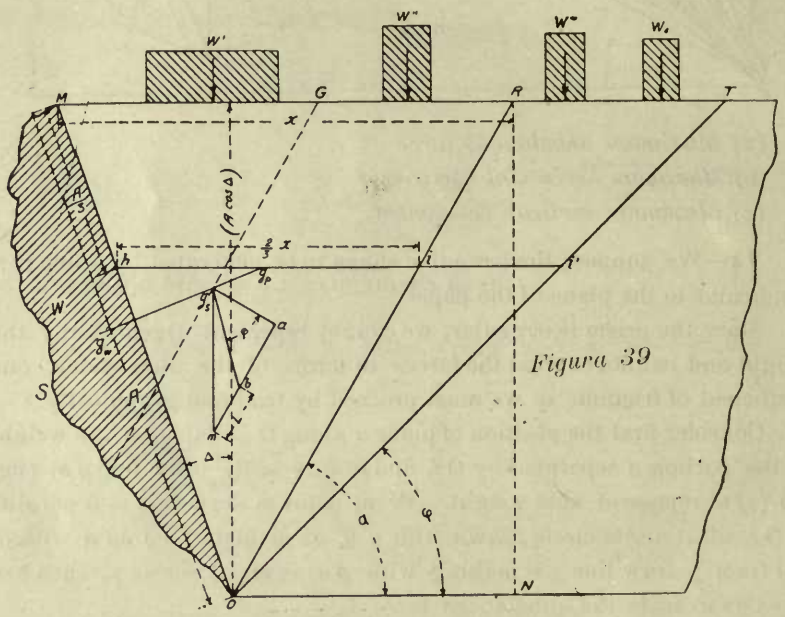
It is obvious that the closer we take the different positions of angle x the nearer will the result be to the true value.

We may solve this problem by calculations, but the method is rather laborious and slow.

Problem N.° 12

75—*Given a mass of an incoherent substance of the shape shown in (figure 39) having an irregular paramento M S O and concentrated loads resting on it, determine the plane α for the following conditions:*

(a) *Maximum unbalanced force.*
(b) *Maximum horizontal component.*
(c) *Maximum vertical component».*



First Condition—Maximum Unbalanced Force

76—This problem has to be solved by trial on account of the irregularity of the paramento and the presence of the concentrated loads.

Consider first only the block, without the loads on it.

Join O with M, M being on the horizontal surface as shewn in the figure.

We have now the part $M\ S\ O$, cut out by $M\ O$, and the prism $R\ M\ O$. Let g_w and g_p be the c. gs of thesse two masses and g_s the resultant centre of gravity of them.

The general equation of the unbalanced force is:

$$L = W (\sin \alpha - \cos \alpha \tan \varphi)$$

where, W = Weight of $M\ S\ O$ + Weight of prism $R\ M\ O$

$$= W + \frac{A\ \gamma\ x \cos \triangle}{2}$$

Now,

$$\sin \alpha = \frac{A \cos \triangle}{\sqrt{A^2 - 2\,A\,x \sin \triangle + x^2}}$$

and

$$\cos \alpha = \frac{x - A \sin \triangle}{\sqrt{A^2 - 2\,A\,x \sin \triangle + x^2}}$$

Therefore;

$$L = \frac{1}{2 \cos \varphi} \times$$

$$\left(\frac{2\,A\,W \cos(\triangle - \varphi) \cos \varphi + x\,(A^2\,\gamma \cos(\triangle - \varphi) \cos \triangle - W \sin(2\varphi)) - x^2\,A\,\gamma \cos \triangle \sin \varphi}{\sqrt{A^2 - 2\,A\,x \sin \triangle + x^2}} \right)$$

Now. calling;

$$x = \omega\,A,\ A^2\,\gamma \cos(\triangle - \varphi) \cos \triangle - W \sin(2\varphi) = B,\ A^2\,\gamma \cos \triangle \sin \varphi = C$$

$$2\,W \cos(\triangle - \varphi) \cos \varphi = F,\ \sin \triangle = H.$$

we have;

$$L = \frac{1}{2 \cos \varphi} \left(\frac{\omega\,B - \omega^2\,C + F}{\sqrt{1 - 2\,\omega\,H + \omega^2}} \right)$$

Differentiating and simplifying we get;

$$\frac{d\,L}{d\,\omega} = \frac{1}{2 \cos \varphi} \left(\frac{-\omega^3\,C + 3\,\omega^2\,C\,H - \omega\,(B\,H + F + 2\,C) + F\,H + B}{(1 - 2\,\omega\,H + \omega^2)\sqrt{A^2 - 2\,\omega\,H + \omega^2}} \right)$$

and equating to zero for a maximum we have;

$$-\omega^3\,C + 3\,\omega^2\,C\,H - \omega\,(B\,H + F + 2\,C) + F\,H + B = 0$$

This equation can be solved by trial as explained in paragraph **11**, or algebraically.

To get rid of term in ω^2, we may do as follows;

$$\omega^3 - 3\,\omega^2\,H + \omega \left(\frac{B\,H + F + 2\,C}{C} - \frac{F\,H + F}{C} = 0 \right.$$

Calling;

$$3H = m, \quad \frac{BH + F + 2C}{C} = S, \quad \frac{FH + B}{C} = T,$$

(1) $\omega^3 - \omega^2 m + \omega S - T = 0.$

Now let $\omega = y + K$ and replace its value in (1).

Then,

$$y^3 + y(S - 3\sin^2\Delta) + S\sin\Delta - 2\sin^3\Delta - T = 0$$

which can be solved in the ordinary way, as explained in paragraph **11** pag. 16, or algebraically.

Also solve for constant K.

The value of K for this case is, $K = \sin\Delta$.

In this way value of x is found and angle α for the maximum unbalanced force is obtained from equation.

$$\tan\alpha = \frac{A\cos\Delta}{x - A\sin\Delta}$$

77. Note—The value of α calculated in the first trial is only for the masses *M S O* and *R M O*.

The next step is to take with these masses load *W'*, and find angle α for the maximum unbalanced force. Do the same, each time including a new load, until position of *O T* is reached.

For each calculation we get a maximun unbalanced force, and out of all these, take the maximum of the maximums and determine angle α for this value.

Second Condition—Maximum Horizontal Component

78—The maximum horizontal component for the unbalanced force is L cos α.

Now,

$$L = \frac{1}{2\cos\varphi}\left(\frac{\begin{array}{c}2AW\cos(\Delta - \varphi)\cos\varphi + xA^2\gamma\cos(\Delta - \varphi)\cos\Delta \\ - xW\sin(2\varphi) - x^2A\gamma\cos\Delta\sin\varphi\end{array}}{\sqrt{A^2 - 2Ax\sin\Delta + x^2}}\right)$$

and

$$\cos \alpha = \frac{x - A \sin \triangle}{\sqrt{A^2 - 2Ax \sin \triangle + x^2}}$$

Therefore;

$$L \cos \alpha = \frac{1}{2 \cos \varphi} \times$$

$$\frac{\left(\begin{array}{l} + A x (2 W \cos (\triangle - \varphi) \cos \varphi - A^2 \gamma \cos (\triangle - \varphi) \sin \Delta \cos \Delta) \\ \qquad + A x W \sin (2 \varphi) \sin \Delta \\ + x^2 (A^2 \gamma \sin \Delta \cos \Delta \sin \varphi + A^2 \gamma \cos (\Delta - \varphi) \cos \Delta - W \sin (2 \varphi)) \\ - x^3 A^2 \gamma \cos \Delta \sin \varphi - 2 A^2 W \cos (\triangle - \varphi) \cos \varphi \sin \triangle \end{array}\right)}{A^2 - 2 A x \sin \triangle + x^2}$$

Now calling;

$x = \omega A$

$2 W \cos (\triangle - \varphi) \cos \varphi - A^2 \gamma \cos (\triangle - \varphi) \sin \triangle \cos \triangle$
$+ W \sin (2 \varphi) \sin \triangle = B$

$A^2 \gamma \sin \triangle \cos \triangle \sin \varphi + A^2 \gamma \cos (\triangle - \varphi) \cos \triangle - W \sin (2 \varphi) = C$

$A^3 \gamma \cos \triangle \sin \varphi \ldots\ldots = F$

$2 W \cos (\triangle - \varphi) \cos \varphi \sin \triangle \ldots\ldots = D$

$\sin \triangle \ldots\ldots = H$

we have;

$$L \cos \alpha = \frac{1}{2 \cos \varphi} \left(\frac{\omega B + \omega^2 C - \omega^3 F - D}{1 - 2 \omega H + \omega^2} \right)$$

Differentiating and simplifying we obtain;

$$\frac{d (L \cos \alpha)}{d \omega} = \frac{1}{2 \cos \varphi} \times$$

$$\left(\frac{-\omega^4 F + 4 \omega^3 F H - \omega^2 (2 C H + B - 3 F) + 2 \omega (C + D) - 2 D H + B}{(1 - 2 \omega H + \omega^2)^2} \right)$$

and equating to zero for a maximum we get;

$$-\omega^4 F + 4 \omega^3 F H - \omega^2 (2 C H + B - 3 F) + 2 \omega (C + D) - 2 D H + B = 0$$

Solve this equation as explained in paragraph **11**, and to find angle α substitute the calculated value of x in equation;

$$\tan \alpha = \frac{x \cos \triangle}{x - A \sin \triangle}$$

The value of α calculated above is for the two masses M S O and R M O.

For the succeeding stages of this problem see special note in paragraph **77**.

Third Condition.—Maximum Vertical Component.

79—The vertical component of the unbalanced force is, $L \sin \alpha$;

Now;

$$L = \frac{1}{2 \cos \varphi} \times$$

$$\left(\frac{2 A W \cos (\triangle - \varphi) \cos \varphi + x (A^2 \gamma \cos (\triangle - \varphi) \cos \triangle - W \sin (2 \varphi) - x^2 A \gamma \cos \triangle \sin \varphi}{\sqrt{A^2 - 2 A x \sin \triangle + x^2}} \right)$$

and

$$\sin \alpha = \frac{A \cos \triangle}{\sqrt{A^2 - 2 A x \sin \triangle + x^2}}$$

Therefore;

$$L \sin \alpha = \frac{A \cos \triangle}{2 \cos \varphi} \times$$

$$\left(\frac{2 A W \cos (\triangle - \varphi) \cos \varphi + x (A^2 \gamma \cos (\triangle - \varphi) \cos \triangle - W \sin (2 \varphi) - x^2 A \gamma \cos \triangle \sin \varphi}{A^2 - 2 A x \sin \triangle + x^2} \right)$$

Now, calling;

$$x = \omega A$$

$$A^2 \gamma \cos (\triangle - \varphi) \cos \Delta - W \sin (2 \varphi) = B$$
$$A^2 \gamma \cos \Delta \sin \varphi \ldots\ldots = C$$
$$2 W \cos (\triangle - \varphi) \cos \varphi \ldots\ldots = F$$
$$\sin \triangle \ldots\ldots = H$$

we have;

$$L \sin \alpha = \frac{\cos \triangle}{2 \cos \varphi} \left(\frac{\omega B - \omega^2 C + F}{1 - 2 \omega H + \omega^2} \right)$$

Differentiating and simplifying we get;

$$\frac{d\,(L \sin \alpha)}{d\,\omega} = \frac{\cos \triangle}{2 \cos \varphi} \left(\frac{\omega^2\,(2\,C\,H - B) - 2\,\omega\,(F + C) + 2\,F\,H + B}{(1 - 2\,\omega\,H + \omega^2)^2} \right)$$

and equating to zero for a maximum we obtain;

$$\omega^2\,(2\,C\,H - B) - 2\,\omega\,(F + C) + 2\,F\,H + B = 0$$

Solving this quadratic we get;

$$\omega_1 = \frac{F + C}{2\,C\,H - B} + \sqrt{\left(\frac{F + C}{2\,C\,H - B}\right)^2 - \frac{2\,F\,H + B}{2\,C\,H - B}}$$

$$\omega_2 = \frac{F + C}{2\,C\,H - B} - \sqrt{\left(\frac{F + C}{2\,C\,H - B}\right)^2 - \frac{2\,F\,H + B}{2\,C\,H - B}}$$

Out of these two values, the one that makes the second differential coefficient negative, gives us the maximum value of L. Angle α is obtained by substituting value of x in the equation,

$$\tan \alpha = \frac{A \cos \triangle}{x - A \sin \triangle}$$

For the succeding stages of this problem see special note on paragraph **77**.

In conclusion, we would advise the reader that, for cases such as we are studying, it is not considered practical to use the analytical-trial method as it is very laborious, whereas on the other hand, the graphical method being much easier and sufficiently exact for the purpose, could be employed much more advantageously.

With this final consideration we conclude the study of the incoherent bodies.

CHAPTER VII

LIQUID BODIES

80—Properties—The free surface of any liquid at rest, is always horizontal, on account of the fact that there is no force of friction between their particles. If the surface were inclined by any amount, however small, equilibrium would be destroyed and motion would ensue, since there is not a tangantial force preventing the liquid particles to slide one on the other.

The formulae that we have studied for the incoherent bodies can only be applied to liquids, for those cases in wich the mass considered is bounded by a horizontal plane. Also, since there is no friction in perfect liquids, the angle $\varphi = 0$.

Our study of the liquids, will then be, of the same nature as for the incoherent bodies and the same formulae can be applied provided we make $\varphi = 0$ and the inclination of the surface is considered always horizontal.

In the case of a mass with an irregular paramento having loads resting on a horizontal surface, if applied to liquids, the problem becomes simpler since no loads can rest on a liquid surface.

Let us treat now a few problems connected with liquid substances.

Problem N.° 1

81—*Given a liquid mass at rest with a vertical paramento as shown (in figure 40) determine plane* α *for the following conditions:*

(a) *Maximum unbalanced force.*
(b) *Maximum horizontal component.*
(c) *Maximum vertical component.*

First Condition—Maximum Unbalanced Force.

82—In Problem N.° 1, page 52, of the incoherent bodies we had for the unbalanced force that;

$$L = \frac{A\,\gamma}{2}\left(\frac{A\,x - x^2 \tan \varphi}{\sqrt{A^2 + x^2}}\right)$$

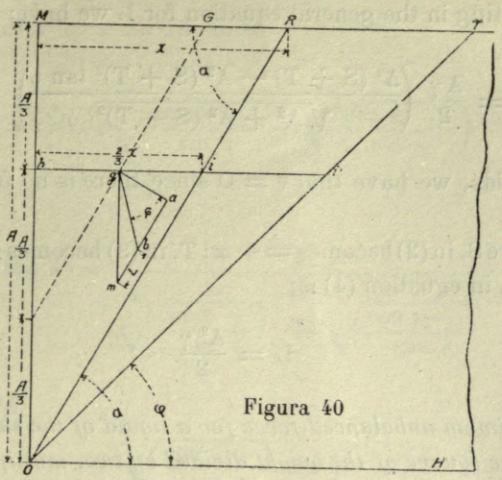


Figura 40

The value of x that makes the above equation a maximum is;

(1) $x =$

$$= A\left(\sqrt[3]{\frac{\cot.\varphi}{2}+\sqrt{\frac{\cot^2\varphi}{4}+\frac{8}{27}}} + \sqrt[3]{\frac{\cot.\varphi}{2}-\sqrt{\frac{\cot^2\varphi}{4}+\frac{8}{27}}}\right)$$

and the tangent of the angle α corresponding to this maximum is;

$\tan \alpha =$

$$= \frac{1}{\sqrt[3]{\frac{\cot\varphi}{2}+\sqrt{\frac{\cot^2\varphi}{4}+\frac{8}{27}}} + \sqrt[3]{\frac{\cot\varphi}{2}-\sqrt{\frac{\cot^2\varphi}{4}+\frac{8}{27}}}}$$

Now, let:

$$(2)\quad \sqrt[3]{\frac{\cot\varphi}{2}+\sqrt{\frac{\cot^2\varphi}{4}+\frac{8}{27}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots = S$$

$$(3)\quad \text{and} \quad \sqrt[3]{\frac{\cot\varphi}{2}-\sqrt{\frac{\cot^2\varphi}{4}+\frac{8}{27}}} =$$

$$= \sqrt[3]{\frac{\cot\varphi}{2}-\frac{\cot\varphi}{2}\sqrt[3]{1+\frac{4}{\cot^2\varphi}\times\frac{8}{27}}} = T$$

then substituting in the general equation for L we have;

$$(4) \qquad L = \frac{A\gamma}{2}\left(\frac{A^2(S+T) - A^2(S+T)^2 \tan\varphi}{\sqrt{A^2 + A^2(S+T)^2}}\right)$$

For liquids, we have that $\varphi = 0$ since there is no friction between the particles.

Therefore S, in (2) becomes $= +\infty$; T, in (3) becomes 0; and $\tan\varphi = 0$, so value of L in equation (4) is;

$$L = \frac{A^2\gamma}{2}$$

i. e. *the maximum unbalanced force for a liquid of the shape considered, is equal to the square of the height divided by two, multiplied by the specific gravity.*

If the liquid considered is water, the specific gravity, being one, we have that;

$$L = \frac{A^2}{2}$$

which agrees with the formula used in hydrostatic to determine the total presure on a vertical surface submerged in water.

The plane of minimum resistance is given by;

$$\tan\alpha = \frac{1}{S+T}$$

Now since $\varphi = 0$, S, becomes $= +\infty$, and $T = 0$.

Therefore, $\tan\alpha = 0$.

and angle $\alpha = 0$.

Consequently the maximum pressure is horizontal, and value of x in equation (1) (the only real one) becomes $+\infty$.

Second Condition.—Maximum Horizontal Component.

83—The horizontal component is, $L\cos\alpha$.

now;

$$(1) \qquad L\cos\alpha = \frac{A\gamma}{2}\left(\frac{A x^2 - x^3\tan\varphi}{A^2 + x^2}\right)$$

The value of x for a maximum is;

$$x = A\left(\sqrt[3]{\frac{\cos\varphi + 1}{\sin\varphi}} + \sqrt[3]{\frac{\cos\varphi - 1}{\sin\varphi}}\right)$$

Therefore;

$$\tan\alpha = \frac{1}{\sqrt[3]{\frac{\cos\varphi - 1}{\sin\varphi}} + \sqrt[3]{\frac{\cos\varphi - 1}{\cos\varphi}}}$$

Now calling;

$$\sqrt{\frac{\cos\varphi + 1}{\sin\varphi}} = S$$

$$\sqrt{\frac{\cos\varphi - 1}{\sin\varphi}} = T$$

and substituting in equation (1) we have;

$$L\cos\alpha = \frac{A\gamma}{2}\left(\frac{A^3(S+T)^2 - A^3(S+T)^3\tan\varphi}{A^2 + A^2(S+T)^2}\right)$$

$$= \frac{A^2\gamma}{2}\left(\frac{1 - (S+T)\tan\varphi}{\frac{1}{(S+T)} + 1}\right)$$

Now, since $\varphi = 0$

S, becomes $= +\infty$

T, becomes $= 0$

So $L\cos\alpha = \frac{A^2\gamma}{2}$

and for water,

$$L\cos\alpha = \frac{A^2}{2}$$

$$\tan\alpha = \frac{1}{S+T} = 0$$

and $$x = A(S+T) = +\infty$$

which is the same result as obtained in Condition 1.

Third Condition.—Maximum Vertical Component.

84—The vertical component is L sin α

$$L \sin \alpha = \frac{A^2 \gamma}{2} \left(\frac{A x - x^2 \tan \varphi)}{A^2 + x^2} \right)$$

The value of x for a maximum is;

$$x = A \left(\frac{1 - \sin \varphi}{\cos \varphi} \right)$$

and for $\varphi = 0$ we have that;

$$x = A.$$

When substituting in, L sin α we get;

$$L \sin \alpha = \frac{A^2 \gamma}{2} \left(\frac{A^2 - A^2 \tan \varphi}{A^2 + A^2} \right) = \frac{A^2 \gamma}{4} \left(1 - \tan \varphi \right)$$

and since $\varphi = 0$, $\tan \varphi = 0$

Therefore;

$$L \sin \alpha = \frac{A^2 \gamma}{4}$$

and for water;

$$L \sin \alpha = \frac{A^2}{4}$$

Now;

$$\tan \alpha = \frac{\cos \varphi}{1 - \sin \varphi} = 1$$

Therefore, $\alpha = 45^\circ$

i.e., for a maximum vertical componet of the unbalanced force, the angle made by the plane of minimum resistance with the horizontal is 45°.

Let us explain the meaning of the vertical component $\frac{A^2}{4}$ just calculated in the case of liquids.

85—Here we had better inform the readers that the term liquids as applied in this book to designate some substances in not exactly the

right one in use. The one which should be used is «*solid substances behaving like perfect liquids in the way they exert pressures*» but this term, liquids has been used simply to avoid long and unecessary explanations.

The reader may judge from that already referred to and which follows what relation mechanically considered implies the definition of liquids to those substances to which they do no pertain.

We are going to show now, the difference in the way of exerting pressure by a frictionless solid and by a frictionless liquid and as an example we select for the purpose, water in the solid and liquid states.

Consider 2 equal prisms *R. M. O.*, (figure 41) (I and II) of an incoherent material shown in figure. Let △ be the angle made by the paramento with the plane of minimum resistance R. O.

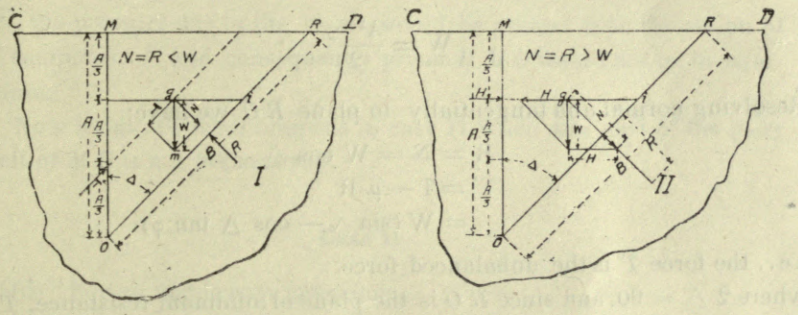


Figura 41

In case I considerer the prism *R. M. Q* with the part of the body to the left of the paramento *M O* taken away.

In case II we take the prism *R M O* inside the whole block and imagine a vertical plane *M O* in the block.

Let us study the pressures in cases I and II for the following conditions:

Case I.

(a) Prisms *R M O* and mass *D R O* of an incoherent substance.
(b) *R M O* and *D R O* for ice.
(c) Prism *R M O* of ice and mass *D R O* of water.

Case II.

(d) Prism *R M O* and mass *D R O* of an incoherent substance.
(e) Prism *R M O* for ice and mass *D R O* for water.

Case I

(a) Let A be the height of the paramento MO.

The weight W of the prism acts at its centre of gravity as shown.

Resolve this weight in two components, N normal to RO, an T tangential to it.

Since we have taken away the part of the body to the left of MO, to keep the prism RMO in equilibrium we have to apply a force T parallel to plane R O.

The forces keeping the prism in equilibrium are;

W. T and reation R.

To simplify this problem, let angle $\triangle$, made by the paramento with the plane of minimum resistance be 45°.

Then, the weight of the prism RMO is;

$$W = \frac{A^2 \gamma}{2}$$

Resolving normal and tangentially to plane RO we have;

$$R = N = W \cos \triangle$$
$$T' = T - \mu R$$
$$= W (\sin \triangle - \cos \Delta \tan \varphi)$$

i.e., the force T' is the unbalanced force.

where $2 \triangle = 90$, and since RO is the plane of minimum resistance, T' is the maximum unbalanced force.

So by applying the force T' inclined to the horizontal at an angle $\triangle$ the prism RMO is kept in equilibrium.

(b). Let us see what happes in this case.

Since prism RMO is made of ice, the angle of friction $\varphi = 0$.

The forces acting for this case are the same as in case (a) except that the force of friction is zero.

So we have,

$$R = N = W \cos \triangle$$
$$T = T'$$
$$\text{i,e, } = W \sin \triangle$$
$$\text{i.e.} = \frac{A^2 \gamma}{2} \times \frac{1}{\sqrt{2}}$$
$$\text{i. e.} = \frac{A^2}{2\sqrt{2}} \qquad \text{if } \gamma = 1$$

So in the case of the ice, the unbulanced force T' to keep the prism in equilibrium is equal and opposite to the component T of the weight tangential to plane $R\,O$ and the reaction of the mass of ice $D\,R\,O$ on plane $R\ O$ is equal to the normal component of the weight of the prism.

(c) Now, suppose that the mass $D\,R\,O$ is converted into liquid and prism $R\,M\,O$ is still of ice. In this case the pressure due to the water on plane $O\,R$ would be

$$R = \frac{A^2}{2}\sqrt{2}$$

and force N would still be $= \dfrac{A^2}{2\sqrt{2}}$

ti. e., the pressure due to the water would be greater than the action of the component N aud consequently prism $R\,M\,O$ would not be in equilibrium.

Now let us see what happens in case 11 when the part of the body to left of $M\,O$ is not taken away.

Case II.

(d) For an incoherent substance.

In this case we shall have a normal pressure H acting on $M\,O$ and another normal N to surface $R\,O$.

Then resolving the weight of the prism normal to surfacen $M\,O$ and $R\,O$ we have;

$$H = W = \frac{A^2\gamma}{2}$$

$$N = \sqrt{2}\ W = \sqrt{2}\ \frac{\gamma A^2}{2}$$

The component H is counterbalanced by the reaction H' on plane $M\,O$, $+\ \mu N$ and component N is opposed by reaction R on plane $R\,O$, thus keeping the prism $R\,M\,O$ in equilibrium, inside the whole mass of the incoherent substance.

The same would happen it the whole block were made of ice.

(e) Now suppose the mass underneath plane $R\,O$ is made of water. In this case R would be equal to the water pressure on surface $R\,O$.

This pressure is.

$$R = \frac{A^2}{2}\sqrt{2}$$

$$\text{But N is also} = \frac{A^2}{2}\sqrt{2}$$

Therefore $R = N$

i. e., there is equilibrium for the prism *R M O*.

So we may conclude that; «*for frictioneless solids and incoherent substances the normal reaction on plane R O is equal and opposite to the normal component of the weight of the prism on that plane, and for liquids this reaction is constant, being for* $\triangle = 45°$,

$$= \frac{A^2}{2}\sqrt{2}$$

When we made zero the coefficient of friction in the formulae that we found for the incoherent substances, to apply them to the liquids. we did not take into account at all, the way the liquids exert their pressures, and consequently the equation L sin α seemed to be in contradiction with the hydrostatic principles.

The formula of the unbalanced force for liquids is obtained directly from that of an incoherent substance, by making the coefficient of friction zero, without taking into consideration the way liquids exert their pressures on a plane surfacce. This means that ice exert normal pressures on plane surface in the same way as water does. Now we have shown that these equations can be applied to ice, consequently they can also be applied to water in the case of the normal unbalanced force, which is the same as the normal water pressure on a plane immersed.

In the following problems we shall find the formulae for the maximum vertical component of the unbalanced force, but we remind the reader that those formulae are to be applied to frictionless solids and not to liquids, since the way they exert pressures is somewhat different.

Problem N.° 2

86. *Given a mass of a liquid body, of the shape shown in (figure 42) having a paramento inclined to the horizontal at an angle* $\triangle$, *determine the position of the plane* α *for the following conditions:*

(a) *Maximum unbalance force.*

(b) *Maximum horizontal component.*

(c) *Maximum vertical component.*

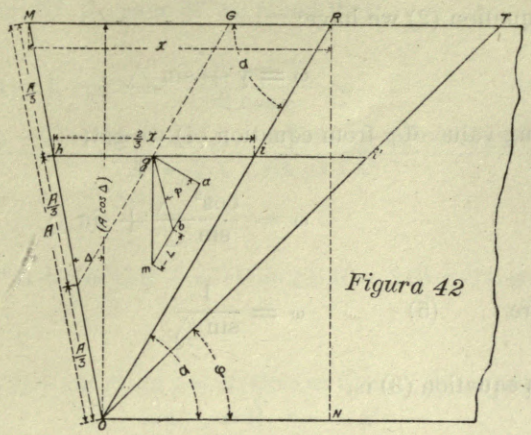


Figura 42

First Condition.—Maximum Unbalanced Force.

87—The general equation of the unbalanced force for a prism of the dimensions shown in figure 41 is;

$$L = \frac{A\ \gamma\ \cos\ \triangle}{2\ \cos\ \varphi}\left(\frac{A\ x\ \cos\ (\triangle - \varphi) - x^2\ \sin\ \varphi)}{\sqrt{A^2 - 2\ A\ x\ \sin\ \triangle + x^2}}\right)$$

In Problem No. 2 of the incoherent bodies, after equating the first differential coefficient of L to zero to find its maximum value, we had that;

$$y^3 + \frac{y}{2}\ (4\ \cos^2\ \Delta\ + \sin\ (2\ \triangle)\ \cot\ \varphi)\ -\ \frac{\cos^2\ \triangle\ \cos\ (\triangle + \varphi)}{\sin\ \varphi}\ =\ 0$$

(1) i. e., $y^3\ \sin\ \varphi + \frac{y}{2}\ \sin\ \varphi\ (4\ \cos^2\ \triangle + \sin\ (2\ \triangle)\ \cot\ \varphi)$

$$-\ \cos^2\ \triangle\ \cos\ (\triangle + \varphi) = 0$$

where;

(2) $y = \omega + k.$

(3) $\omega = \frac{x}{A}$

and $k = \sin\ \triangle$

In the case of liquids, making angle $\varphi = 0$ equation (1) becomes.

(4) $y\ \sin\ \triangle\ -\ \cos^2\ \triangle = 0$

From equation (2) we have;

$$\omega = y + \sin \triangle$$

Replacing value of y from equation (4) we get;

$$\omega = \frac{\cos^2 \triangle}{\sin \triangle} + \sin \triangle$$

Therefore (5) $$\omega = \frac{1}{\sin \triangle}$$

then, x from equation (3) is,

(6) $$x = \frac{A}{\sin \triangle}$$

Now substituting the above value in (1) we have;

(7) $$L = \frac{A^2 \gamma \cos \triangle}{2}$$

and in (2), we obtain.

(8) $$\tan \alpha = \tan \triangle$$

Equation (8) tells us, that the maximum value of the unbalanced force acts normal to the paramento $M\ O$ and equation (7) gives the value of this maximum pressure.

This pressure is equal to the length A of the paramento, times the vertical projection, divided by 2, multiplied by the specific gravity of the liquid.

For water, equation (7) becomes;

(9) $$L = \frac{A^2 \cos \triangle}{2}$$

which agrees with the formula given in hydrostatic for the pressure on a submerged plane surface inclined at an angle $\triangle$ with the vertical.

Second Condition.—Maximum Horizontal Component.

88—The horizontal component of the pressure on surface $M\ O$ is L cos α.

In Problem N°. 2., page 57 we found that;

$$(1)\quad \mathrm{L}\cos\alpha = \frac{\mathrm{A}\,\gamma\cos\triangle}{2\cos\varphi}\times \left(\frac{\mathrm{A}\,x^2(\cos(\triangle-\varphi)+\sin\triangle\sin\varphi)-x^3\sin\varphi-\mathrm{A}^2 x\cos(\triangle-\varphi)\sin\triangle}{\mathrm{A}^2-2\,\mathrm{A}\,x\sin\triangle)+x^2}\right)$$

and that;

$$(2)\quad -\omega^4\,\mathrm{C}+4\,\omega^3\,\mathrm{C\,H}+\omega^2(\mathrm{F}-2\,\mathrm{B\,H}-3\,\mathrm{C})+2\,\omega\,\mathrm{B}-\mathrm{F}=0$$

where;

$$\cos(\triangle-\varphi)+\sin\triangle\sin\varphi=\mathrm{B},\ \sin\varphi=\mathrm{C},\ \cos(\triangle-\varphi)\sin\triangle=\mathrm{F}$$
$$\sin\triangle=\mathrm{H},\ x=\omega\,\mathrm{A}$$

then, making $\varphi=0$ we have that the above values become;

$$\cos\triangle=\mathrm{B},\ 0=\mathrm{C},\ \cos\triangle\ \text{sín}\ \triangle=\mathrm{F},\ \sin\triangle=\mathrm{H},\ x=\omega\,\mathrm{A}$$

Then substituting in equation (2) their values, we get;

$$\omega^2(\sin\triangle\cos\triangle-2\cos\triangle\sin\triangle)+2\,\omega\cos\triangle-\cos\triangle\sin\triangle=0$$

Solving this equation we have;

$$\omega_1=\frac{1-\cos\triangle}{\sin\triangle}$$

$$\omega_2=\frac{1+\cos\triangle}{\sin\triangle}$$

Then since $x=\omega\,\mathrm{A}$, we have;

$$x_1=\mathrm{A}\,\frac{1-\cos\triangle}{\sin\triangle}$$

Terefore, substituting value of x in (1) and considering $\varphi=0$, we get;

$$(3)\qquad \mathrm{L}\cos\alpha=\frac{\mathrm{A}^2\,\gamma\cos\triangle}{4}\left(1+\cos\triangle\right)$$

and;

$$\tan\alpha=\frac{\sin\triangle}{1+\cos\triangle}$$

For equation (3) see paragraph **85**.

Third Condition.—Maximum Vertical Component.

89—The vertical component of the pressure on surface *M O* is L sin α.

In problem N.° 2, page 58 we found that;

$$(1) \qquad L \sin \alpha = \frac{A^2 \gamma \cos^2 \triangle}{2 \cos \varphi} \left(\frac{A x \cos (\triangle - \varphi) - x^2 \sin \varphi}{A^2 - 2 A x \sin \triangle + x^2} \right)$$

also that;

$$\cos (\Delta - \varphi) = B$$

$$(2) \qquad \sin \varphi \ldots\ldots = C$$

$$\sin \triangle \ldots\ldots = H$$

$$(3) \qquad \text{and, } x = \omega A$$

The value of ω giving the maximum vertical component of L for an incoherent body is obtained from equation;

$$(4) \qquad \omega^2 (2 C H - B) - 2 \omega C + B = 0$$

and angle α is given by

$$(5) \qquad \tan \alpha = \frac{A \cos \triangle}{x - A \sin \triangle}$$

Now for liquids $\varphi = 0$

Therefore equations in (2) become;

$$\cos \triangle = B$$

$$0 \quad = C$$

$$\sin \triangle = H$$

and substituting these values in equation (4) we have;

$$- \omega^2 \cos \Delta + \cos \triangle = 0$$

$$\text{i. e., } \cos \triangle (1 - \omega^2) = 0$$

therefore;

$$\omega = \pm 1$$

and x in (3) becomes;

$$x = \pm A$$

consequently L sin α by substituting in (1) would be;

$$(6) \qquad L \sin \alpha = \frac{A^2 \gamma \cos^3 \triangle}{4(1 - \sin \triangle)}$$

and the tangent of the angle α in (5) becomes by substituting value of x;

$$\tan \alpha = \frac{\cos \triangle}{1 - \sin \triangle}$$

For equation (6), see paragraph **85**.

Problem No. 3

90—*Given a mass of a liquid body having a paramento inclined to the vertical at an angle* $\triangle$ *(figure 43) determine the position of plane* α *for the following conditions:*

(a) *Maximum unbalanced force.*
(b) *Maximum horizontal component.*
(c) *Maximum vertical component».*

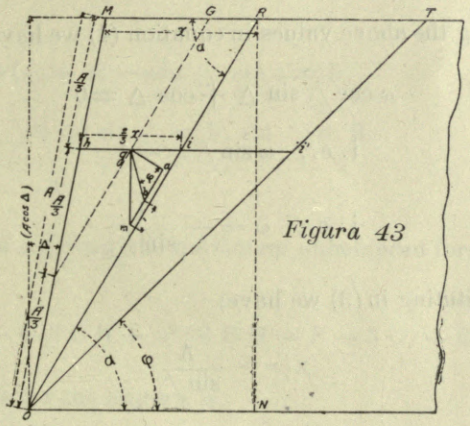


Figura 43

First Condition—Maximum Unbalanced Force

91—The unbalanced force for a mass of an incoherent body of the shape shown is;

$$(1) \qquad L = \frac{A^2 \gamma \cos \triangle}{2 \cos \varphi} \left(\frac{\omega B - \omega^2 C}{1 + 2 \omega H + \omega^2} \right)$$

(see problem No. 3, pag. 59).

where;

$$(2) \qquad \begin{cases} \cos(\triangle + \varphi) = B \\ \sin \varphi \ldots\ldots = C \\ \sin \triangle \ldots\ldots = H \end{cases}$$

$$(3) \qquad \text{and } x = \omega A$$

The value of ω giving the maximum unbalanced force is given by equation;

$$(4) \qquad -\omega^3 C - 3\omega^2 C H + \omega (B H - 2 C) + B = 0$$

The equation for the angle α is,

$$(5) \qquad \tan \alpha = \frac{A \cos \triangle}{x + A \sin \triangle}$$

Por liquids, $\varphi = 0$

Then equations in (2) become,

$$\cos \triangle = B \qquad 0 = C, \qquad \sin \triangle = H$$

Substituting the above values in equation (4) we have;

$$\omega \cos \triangle \sin \triangle + \cos \triangle = 0$$

$$\text{i. e., } \omega \sin \triangle = -1$$

$$\text{i. e., } \omega = -\frac{1}{\sin \triangle}$$

Therefore substituting in (3) we have;

$$x = -\frac{A}{\sin \triangle}$$

The value of the maximum unbalanced force for liquids would then be,

$$L = -\frac{A^2 \gamma \cos \triangle}{2}$$

substituting values in equation (1).

The value of the tangent of α given by (5) becomes;

$$\tan \alpha = \frac{A \cos \triangle}{A \sin \triangle - \frac{A}{\sin \triangle}}$$

$$\text{i. e.,} = - \frac{\sin \triangle \cos \triangle}{1 - \sin^2 \triangle}$$

$$\text{i. e.,} = - \tan \triangle$$

Therefore, $\alpha = - \triangle$

The results obtained here are the same as those obtained in condition of previous problem, having opposite sings as would be expected.

Second Condition — Maximum Horizontal Component

92—The horizontal component for a mass of this shape is L cos α,

$$(1)\ L \cos \alpha = \frac{A^2 \gamma \cos \triangle}{2 \cos \alpha} \left(\frac{\omega^2 B - \omega^3 C + \omega F}{1 + 2 \omega H + \omega^2} \right)$$

(see Problem N.° 3, of the incoherent bodies, second condition, page 61)

where

$$(2) \qquad \cos (\triangle + \varphi) - \sin \triangle \sin \varphi = B,$$

$\sin \varphi = C$, $\cos (\triangle + \varphi) \sin \triangle = F$, $\sin \triangle = H$.

(3) and, $x = \omega A$.

The value of ω giving the maximum unbalanced force is given by equation.

$$(4) \quad \omega^4 C - 4 \omega^3 C H + \omega^2 (2 B H - F - 3 C) + 2 \omega B + F = 0$$

The equation for the angle α is.

$$(5)\ \tan \alpha = \frac{A \cos \Delta}{x + A \sin \triangle}$$

Por liquids $\varphi = 0$.

Then equations in (2) become

$$\cos \triangle = B,\ 0 = C,\ \cos \triangle \sin \triangle = F,\ \sin \triangle = H$$

substituting the above values in equation (4) we have;

$$\omega^2 (2 \sin \triangle \cos \triangle - \sin \triangle \cos \triangle) + 2\omega \cos \triangle + \sin \triangle \cos \triangle = 0$$

$$\text{i. e., } \omega^2 \sin \triangle \cos \triangle + 2\omega \cos \triangle + \sin \triangle \cos \triangle = 0$$

whose roots are;

$$\omega_1 = - \left(\frac{1 + \cos \triangle}{\sin \triangle} \right)$$

$$\omega_2 = - \left(\frac{1 - \cos \triangle}{\sin \triangle} \right)$$

Then equation (1) becomes;

$$L \cos \alpha = - \frac{A^2 \gamma \cos \triangle}{4} \left(1 + \cos \triangle \right)$$

and equation (5) would be;

$$\tan \alpha = - \left(\frac{\sin \Delta}{1 + \cos \Delta} \right)$$

The results obtained here are the same to those obtained in Condition 2, of the previous problem, but with opposite signs.

Third Condition.—Maximum Vertical Component.

93—The vertical component for a mass of this shape is, L sin α;

$$(1) \quad L \sin \alpha = \frac{A^2 \gamma \cos^2 \Delta}{2 \cos \varphi} \left(\frac{\omega B - \omega^2 C}{1 + 2\omega H + \omega^2} \right)$$

(see Problem No. 3, of the incoherent bodies, 3rd, Condition, page 62)

where,

$$(2) \quad \cos (\triangle + \varphi) = B, \sin \varphi = C, \sin \triangle = H.$$

$$(3) \quad \text{and, } \omega = \frac{X}{A}$$

The value of ω giving the maximum unbalanced force is given by

$$(4) \quad - \omega^2 (2 C H + B) - 2 \omega C + B = 0.$$

The equation of the angle α is

$$(5)\ \tan \alpha = \frac{A \cos \Delta}{x + A \sin \Delta}$$

For liquids, $\varphi = 0$.

Then equation in (2) becomes

$$\cos \Delta = B,\ 0 = C,\ \sin \Delta = H.$$

Substituting the above values in (4) we have

$$-\omega^2 \cos \Delta + \cos \Delta = 0.$$

$$\text{i. e.,}\quad \omega = \pm 1.$$

Therefore substituting in (3) we have.

$$A = \pm x$$

and substituting in (1) we get:

$$L \sin \alpha = \frac{A^2 \cos^2 \Delta}{2} \times \frac{\cos \Delta}{2(1 - \sin \Delta)}$$

$$\text{i. e.;} = \frac{A^2 \cos^3 \Delta}{4\ (1 - \sin \Delta)}$$

Putting value of x in (5) we obtain;

$$\tan \alpha = - \frac{\cos \Delta}{1 - \sin \Delta}$$

94—To show how the force $L = - \frac{A^2 \gamma \cos \Delta}{2}$ (see this equation in condition 1) is acting, consider the prisms *O M R.* (Problem N.° 2) and *O M R.* (Problem N.° 3) resting one on the other forming a whole block *K O E* as shown in (figure 44).

Draw *K O* and *E O* making angle Δ with the horizontal (see equation (8) in Problem N. 2, first condition).

Then,

E V = *K V.*

V being on the vertical through point *O.*

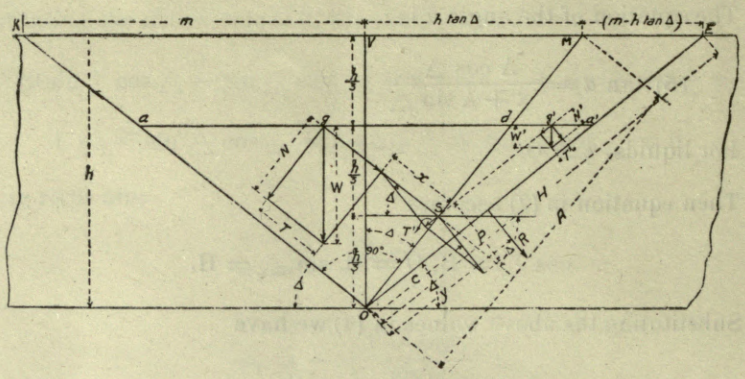


Figura 44

The forces keeping the two prisms in equilibrium are W, W' and the reactions on planes $K\,O$ and $E\,O$; W and W' being the weights of the prisms acting at the their respective c. gs as in figure.

$$\text{Now, } W = (m + h \tan \triangle)\frac{h}{2}$$

$$W' = (m - h \tan \triangle)\frac{h}{2}$$

Replacing m by its value

$$m = \frac{h}{\tan \triangle}$$

in the above equations we have,

$$W = \left(\frac{h \cos \triangle}{\sin \triangle} + \frac{h \sin \triangle}{\cos \triangle}\right)\frac{h}{2} = \frac{h^2}{\sin (2 \triangle)}$$

$$W' = \left(\frac{h \cos \triangle}{\sin \triangle} - \frac{h \sin \triangle}{\cos \triangle}\right)\frac{h}{2} = h^2 \cot (2 \triangle)$$

Resolve weights W and W' normal and tangentially to planes $O\,K$ and $O\,E$ respectively.

Let N, T, N' and T' be these components.

Then;

$$T = W \sin \triangle = \frac{h^2}{\sin (2 \triangle)} \sin \triangle = \frac{h^2}{2 \cos \triangle}$$

since $h = A \cos \triangle$

$$T = \frac{A^2 \cos \triangle}{2}$$

i. e., is the unbalanced force for the prism $M\,O\,K$ (see equation **(7)** Problem N.° 2).

$$N = W \cos \triangle = \frac{h^2}{\sin (2 \triangle)} \cos \triangle = \frac{h^2}{2 \sin \triangle} = \frac{A^2 \cos^2 \triangle}{2 \sin \triangle}$$

and is equal to the reaction R on plane $K\,O$.

$$N' = W' \cos \triangle = h^2 \cot (2 \triangle) \cos \triangle = A^2 \cos^3 \triangle \cot (2 \triangle)$$

and is equal to the reaction on plane $O\,E$ due to the weight of the prism $E\,M\,O$ only.

$$T' = W' \sin \triangle = h^2 \cot (2 \triangle) \sin \triangle$$

This force T' acting tangentially to plane $O\,E$ can be resolved in two forces x, and P, normal to $M\,O$ and $O\,E$ respectively.

So resolving T' normal to these planes, we have;

$$T' = x \cos (2 \triangle)$$

Therefore;

$$x = \frac{T'}{\cos (2 \triangle)} = \frac{h^2 \cot (2 \triangle) \sin \triangle}{\cos (2 \triangle)} =$$

$$= \frac{A^2 \cos^2 \triangle \cos (2 \triangle) \sin \triangle}{2 \sin \triangle \cos \triangle \cos (2 \triangle)} = \frac{A^2 \cos \triangle}{2}$$

But;

$$T = \frac{A^2 \cos \triangle}{2}$$

Therefore;

$$T = -x$$

These forces are equal and opposite and consequently value of x agrees with the one obtained for L in Problem No. 3, where we had that;

$$L = -\frac{A^2 \cos \triangle}{2}$$

acting normal to the plane $M\,O$.

The other componente P of T' would be,

$$P = x \sin(2\triangle) = \frac{A^2 \cos \triangle}{2} \times 2 \sin\triangle \cos \triangle = A^2 \cos^2 \triangle \sin \triangle$$

Then, the total reaction on $O\,E$ would be,

$$P + N' = A^2 \cos^2 \triangle \sin \triangle + A^2 \cos^3 \triangle \cot(2\triangle) =$$
$$= A^2 \left(\frac{2\cos^2 \triangle \sin^2 \triangle + \cos^4 \triangle - \cos^2 \triangle \sin^2 \triangle}{2 \sin \triangle} \right) =$$
$$= A^2 \cos^2 \triangle \left(\frac{\sin^2 \triangle + \cos^2 \triangle}{2 \sin \triangle} \right) = \frac{A^2 \cos^2 \triangle}{2 \sin \triangle}$$

whisch is equal to reaction on plane $O\,K$

The distance of point of application of reaction P from O is,

$$c = \frac{A}{3} \sin(2\triangle)$$

and that of reaction N' is

$$c + \frac{H}{3}, \quad \text{from } O$$

Taking moments about O we have;

$$y \times R = P \times c + N' \times \left(c + \frac{H}{3} \right)$$

Substituting an simplifying we get;

$$y = \frac{H}{3}$$

Problem No. 4

95.—*Given a mass of a liquid body having an irregular paramento* O S M *generated by a line keeping parallel to itself in its different positions (figure 45) determine the plane* x *for the following conditions:*

(a) *Maximum unbalanced force.*
(b) *Maximum horizontal component.*
(c) *Maximum vertical component.*

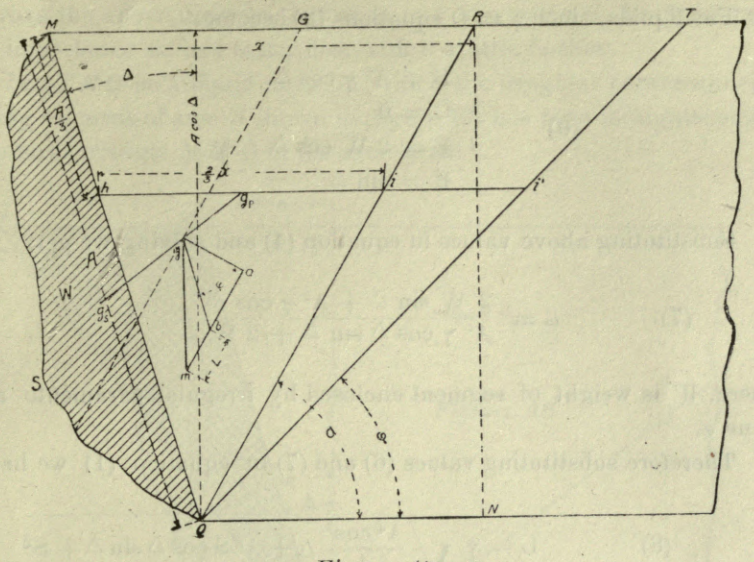


Figura 45

First Condition—Maximum Unbalanced Force

96—The unbalanced force for an incoherent body of the same shape was calculated in Problem No. 12, page 92, to be;

$$(1) \qquad L = \frac{1}{2\cos\varphi}\left(\frac{\omega\ B - \omega^2\ C + F}{1 - 2\ \omega\ H + \omega^2}\right)$$

where;

$$(2) \quad A^2\gamma\cos(\triangle - \varphi)\cos\Delta - W\sin 2\varphi = B, \quad A^2\gamma\cos\triangle\ \sin\varphi = C,$$
$$2\,W\cos(\triangle - \varphi)\cos\varphi = F, \quad \sin\Delta = H$$

$$(3) \qquad \text{and} \qquad x = \omega\,A$$

The equation for ω giving the maximum value of the unbalanced force is;

$$(4) \qquad -\omega^3\,C + 3\,\omega^2\,C\,H - \omega(B\,H + F + 2\,C) + F\,H + B = 0$$

and the tangent of angle α is,

$$(5) \qquad \tan\alpha = \frac{A\cos\triangle}{x - A\sin\triangle}$$

For liquids, since $\varphi = 0$ equations (2) become,

$$(6) \qquad \left.\begin{array}{l} B = A^2 \gamma \cos \Delta \\ C = 0 \\ F = 2\ W \cos \Delta \\ H = \sin \Delta \end{array}\right\}$$

Substituting above values in equation (4) and solving we get;

$$(7) \qquad \omega = \frac{2\ W \sin \Delta + A^2 \gamma \cos \Delta}{A^2 \gamma \cos \Delta \sin \Delta + 2\ W}$$

where, W is weight of segment enclosed by irregular paramento and plane α.

Therefore substituting values (6) and (7) in equation (1) we have;

$$(8) \qquad L = \gamma \sqrt{\frac{A^4 \cos^2}{4} \Delta + A^2\, S \cos \Delta \sin \Delta + S^2}$$

S being the area of the segment of weight W; and equation (5) by substitution becomes;

$$(9) \qquad \tan \alpha = \frac{A^2 \cos \Delta \sin \Delta + 2\ S}{A^2 \cos^2 \Delta}$$

In equations (8) and (9) we see that the values of the unbalanced force and that of the angle α are independent of the shape that the segment $O\ S\ H$ may have, provided the value of S remains constant; so we may enunciate this principle as follows.

97. Theorem.—*If two horizontal planes* A *and* B *bounding a liquid mass in equilibrium, be cut by an inclined plane* C, *and the intersection of this plane with* A *and* B *be considered as the initial and final positions of a line generating a surface* S, *by keeping itself parallel as it moves; the intensity and the direction of the water pressures on the faces of the prism generated is the same, no matter what shape the generating curve may have, provided the area enclosed by it and the plane* C *is constant.*

The equations to prove this theorem are derived from the equation of the unbalanced force;

$$L = W (\sin \alpha - \cos \alpha \tan \varphi)$$

We have shown in Problem N.° **1**, of the coherent bodies, that this equation has a general character, and consequently this fact is enough

to prove the above theorem, but as it is a new principle having a certain importance we had better deal with it a little further.

Let TMO be a liquid mass in which the irregular paramento enclosing segment of area S shown in (figure 46) has been straightened so as to form triangle $MM'O$ of the same area.

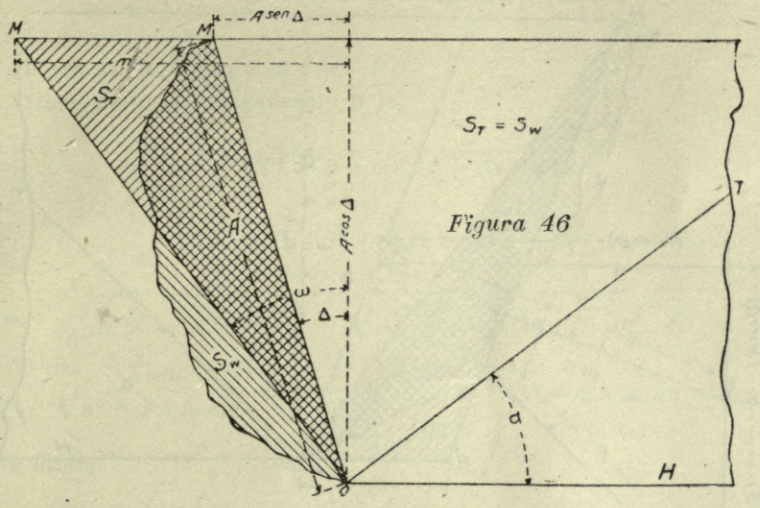


Figura 46

We are going to show that although we change the shape of the irregular paramento, provided we keep the area constant, the pressure due to the prism of liquid remains constant, also that for a maximum pressure $\alpha = \omega$, i.e., normal to MO.

From figure.

$$(10) \qquad m = A \cos \triangle \tan \omega$$

$$(11) \qquad S = \frac{(m - A \sin \triangle) A \cos \Delta}{2}$$

By substituting in (**11**) the value of m obtained in (**10**) we have;

$$(12) \qquad \tan \omega = \frac{A^2 \cos \Delta \sin \triangle + 2S}{A^2 \cos^2 \triangle}$$

which is the same as the values obtained for tan α in equation (**9**):

Therefore, $\alpha = \omega$;

and consequently the unbalanced force;

$$(15) \qquad L = \gamma \sqrt{\frac{A^4 \cos^2 \triangle}{4} + A^2 S \cos \triangle \sin \triangle + S^2}$$

(see equation **8**) has the same value in this case, since area S has remained constant although its shape has been altered.

As a further proof, let us give to the segment S the shape shown in (figure **47**).

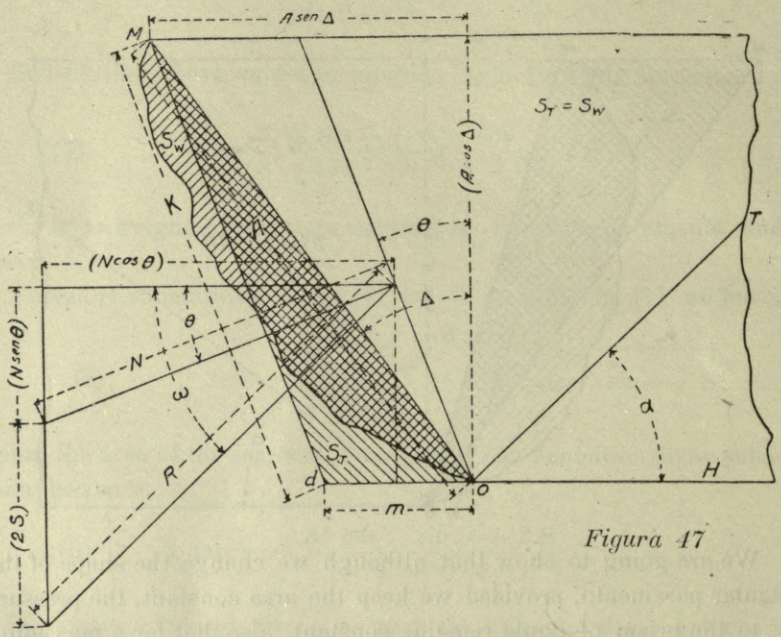


Figura 47

The water pressures on $M\,d$ and $d\,O$ are normal.

Let these pressures be N and $\frac{1}{2}\,S$ and θ angle made by $d\,M$ with the vertical.

From figure we have,

$$A \sin \Delta - m = A \cos \Delta \tan \theta$$

$$\sin \theta = \frac{A \sin \Delta - m}{\sqrt{A^2 - 2\,A\,m \sin \Delta + m^2}} \tag{14}$$

$$\cos \theta = \frac{A \cos \Delta}{\sqrt{A^2 - 2\,A\,m \sin \Delta + m^2}} \tag{15}$$

and;

$$K = \frac{A \cos \Delta}{\cos \theta} = \sqrt{A^2 - 2\,A\,m \sin \Delta + m^2}$$

The water pressure on $M\,d$ is;

$$(16) \qquad N = \frac{K^2 \cos\theta}{2}, \text{ and substituting}$$

$$= \frac{A\ \cos\triangle}{2}\sqrt{A^2 - 2\,A\,m \sin\triangle + m^2}$$

The normal pressure en $d\,O$ is $2\,S$.
Then, the resultant pressure R is;

$$R = \sqrt{(2\,S + N \sin\theta)^2 + (N \sin\theta)^2}$$

$$\text{i. e.} = \sqrt{4\,S^2 + 4\,N\,S \sin\theta + N^2}$$

$$\text{i. e.} \quad \sqrt{4\,S^2 + 4\,S\frac{A\ \cos\triangle}{2}\sqrt{A^2 - 2\,A\,m\sin\triangle + m^2}} \times \sqrt{\frac{A \sin\triangle - m}{\sqrt{A^2 - 2\,A\,m\sin\triangle + m^2}} + \frac{A^2\cos^2\triangle}{4}(A^2 - 2\,A\,m\sin\triangle + m^2)}$$

from figure;

$$m = \frac{2\,S}{A\cos\Delta}$$

Therefore;

$$(17) \qquad R = \gamma\sqrt{\frac{A^4\cos^2\Delta}{4} + A^2\,S\sin\Delta\cos\Delta + S^2}$$

which is the same value as the one we found for the unbalanced force when the paramento was irregular.

From figure we also have;

$$\frac{R}{\cos\theta} = \frac{N}{\cos\omega}$$

$$\text{i. e.} \qquad \cos\omega = \frac{N\cos\theta}{R}$$

Therefore;

$$\tan\alpha = \sqrt{\frac{R^2 - N^2\cos^2\theta}{N\cos\theta}}$$

and substituting values of R, N and cos θ from (17), (16) and (15) we have;

$$\tan \omega = \frac{A^2 \cos \triangle \sin \triangle + 2S}{A^2 \cos^2 \Delta}$$

which is the same as the value obtained for the tangent of α in equation (9) for the irregular paramento.

The two triangles of area S in (figures 46 and 47) have the same base A, and if to these triangles keeping the same base A, we change their shapes without altering their areas, the vertexes will always move along a line parallel to A, so we may state the following;

In any paramento formed by two sides of a triangle having its vertex along a line parallel to the other side, the magnitude and direction of the liquid pressure is constant.

The principle we have just demostrated simplifies considerably the formulae to be used in practice.

In case we have an irregular paramento, we have to solve the following equations;

$$L = \gamma \sqrt{\frac{A^4 \cos^2 \triangle}{4} + A^2 S \cos \triangle \sin \triangle + S^2}$$

and;

$$\tan \alpha = \frac{A^2 \cos \triangle \sin \triangle + 2S}{A^2 \cos^2 \triangle}$$

To solve them we require to know area S, and knowing S we may give the irregular paramento the shape shown in (figure **46**) and apply the equations corresponding to Problem N.° 2, being careful to take, instead of paramento A, the paramento OM shown in the same (figure **46**). Then the equations to be applied in this case would be (*a*) and (**8**) (Problem N.° 2 paragraph **87**).

$$(a) \quad L = \frac{A^2 \gamma \cos \triangle}{2}$$

$$\text{and, } (8) \ \tan \alpha = \tan \triangle$$

With reference to the point of application of the force L, it varies with the change of shape, acting at the new resultant centre of gravity.

If the normal pressure on the paramento is constant whatever the shape may be, so must the horizontal and vertical components.

To find the value of these components, see Conditions **2** and **3** of Problem N.° **12** of the incoherent substances, making zero the coefficient of friction when applying to liquids.

Comparison of Methods

98—To conclude the first part of our work let us take an example and compare the results arrived at, by using Poncelet's, Rankine's and the Author's formulae for an incoherent substance such as earth, having a vertical paramento *M O* and a surface *M T* inclined to the horizon at an angle θ as shown in (figure **48**).

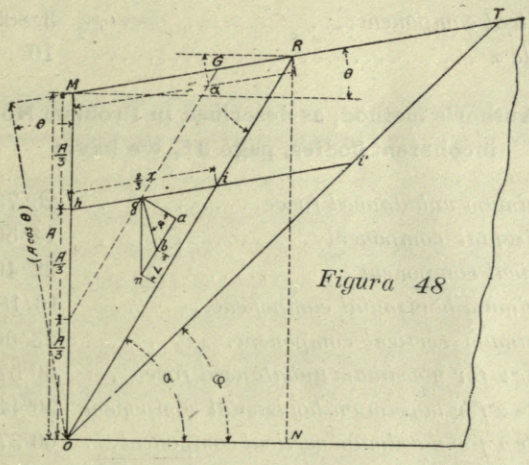


Figura 48

We require to find;

(*1*) *Maximum unbalanced force.*
(*2*) *Horizontal component of the maximum unbalanced force.*
(*3*) *Vertical component of same force.*
(*4*) *Maximum horizontal component.*
(*5*) *Maximum vertical component.*
(*6*) *Angle* α *made by the plane R O with the horizontal.*

Let the mass have the following characteristics;

Height of paramento, A ..... $= 10$ *metres.*
Inclination of surface M T with the horizon, $= 10^0$
Unit weight of earth ..... $= 1.6$ *ton per* m^3
Angle of repose, φ ..... $= 35^0$

Applying Poncelet's graphic method as described by Mr. A. Flamant in page 134 of his work «*Stabilité des Constructions*». Third Edition, we have;

(1) *Maximum unbalanced force*..... 22·100 tons
(2) *Horizontal component*..... 18·000 »
(3) *Vertical component*..... 12·500 »
(4) *Angle* α..... 35°· »

Applying Rankine's analytic method as described in page 251 of «*A Manual of Applied Mechanics*» by William John Macquorn Rankine, 18th edition, 1908, we have;

(1) *Maximum unbalanced force*..... 22·458 tons
(2) *Horizontal component*..... 22·121 »
(3) *Vertical component*..... 3·885 tons
(4) *Angle* α..... 10°

Applying Author's method, as described in Problem No. 5 of the incoherent bodies, page 31, we have;

(1) *Maximum unbalanced force*..... 26'735 »
(2) *Horizontal component*..... 14·569 »
(3) *Vertical component*..... 22·403 »
(4) *Maximum horizontal component*..... 16·187 »
(5) *Maximum vertical component*..... 23·064 »
(6) *Angle* α *for maximum unbalanced force*.... 56°57'
(7) *Angle* α *for maximum horizontal component* 50°44'
(8) *Angle* α *for maximum vertical component*... 60°37'

To facilitate the comparison of the figures, let us tabulate the results as follows:

Comparison of Methods

TABLE I.

METHODS	FORCES IN TONS					ANGLE α		
	MAX. FORCE L	HOR. COMPONENT OF L	VER COMPONENT OF L	MAX. HOR. COMP.	MAX. VERT. COMP.	MAX. FORCE L	MAX. HOR. COMP.	MAX. VERT. COMP.
Poncelet	22·100	18·000	12·500	—	—	35°	—	—
Rankine	22·450	22·121	3·885	—	—	10°	—	—
The Author	26·735	14·569	22·403	16·187	23·064	56°57'	50°44'	60°37'

The difference between the results obtained in Rankine's and the Author's method becomes greater as angle θ decreases, and for $\theta = 0$ in the same example we have in Table II:

Comparison of Methods

TABLE II

METHODS	FORCES IN TONS					AMGLE α		
	MAX. FORCE L	HOR. COMPONENT OF L	VER. COMPONENT OF L	MAX. HOR. COMP.	MAX. VERT. COMT.	MAX. FORCE L	MAX. HOR. COMP.	MAX. VERT COMP'
RANKINE	21·608	21·608	0	——	——	0°	——	——
THE AUTHOR	24·000	12·236	20·560	13·520	20·880	59°02′	51°41′	62°30′

We call the atention of the reader to the general differences in the results obtained by using Rankine's and the Author's formulae. In Table II, the maximum horizontal component according to Rankine is 21.608 tons, whilst the text book gives 13.52 tons. Greater still is the difference for the vertical component, being 0 according to Rankine and 20.88 tons to the Author. The same thing may be said of the angle α for the maximum unbalanced force: Rankine gives 0 and we obtain 59°02′.

Refering to the maximum horizontal component of the unbalanced forces 21.608 and 13.520 tons obtained by the two methods we call the attention of the reader to what Mr. A. Flamant (*Inspecteur Général des Ponts et Chaussées*) says in his work on «*Stabilité des Constructions*», page 144, Third Edition, commenting on Rankine's results. He says textually;

«*Toutefois, des expériences et observations récentes, faites en Angle « terre par M. Darwin et M. Baker, et par M. Gobin en France, ont appelé « l'attention sur le degré d'approximation donné par les formules de Ran- « kine et ont montré que cette approximation était insuffisante. Les va « leurs expérimentales trouvées étaient notablement inférieures à celles que « donnait la théorie et n'en étaient quelquefois que la moitié*».

END OF FIRST PART

RETAINING WALLS

SECOND PART

CHAPTER I

1.—In the first part of this work we have studied the way of calculating the intensity of pressure exerted by an homogeneous mass in equilibrium, on a plane cutting the mass, for coherent, non-coherent and liquid substances. In the second part we shall deal with the design of the walls that can stand the thrust produced by those masses.

INCOHERENT BODIES

2.—Since the degree of coherence of these substances varies a great deal, the maximum height that can be attained by the different kind of bodies will also vary under the same conditions.

For masses of iron, granite or other strongly coherent bodies, the heights that can have before breaking are large (see examples in page 23) and altogether of no practical use. Even if we had a mass of iron of such a height that willd break by its own weight and we had to apply to it a retaining wall, of what material will that wall be made of? The correct thing to do would be to buil that wall giving to it a talud sufficient to prevent rupture, but a case like this will never be met in practice.

Let us take a practical example.

Consider a mass of hardened earth of the shape shown in (figure 49) with a vertical paramento M O. Let A be the maximum height that the mass can have before breaking, with no loads such as W, W′, W″, etc., resting on it. As soon as we place the concentrated loads, rupture will occur along the plane of minimum resistance, and let O R be this plane inclined at angle α to the horizon; to prevent the earth from falling we

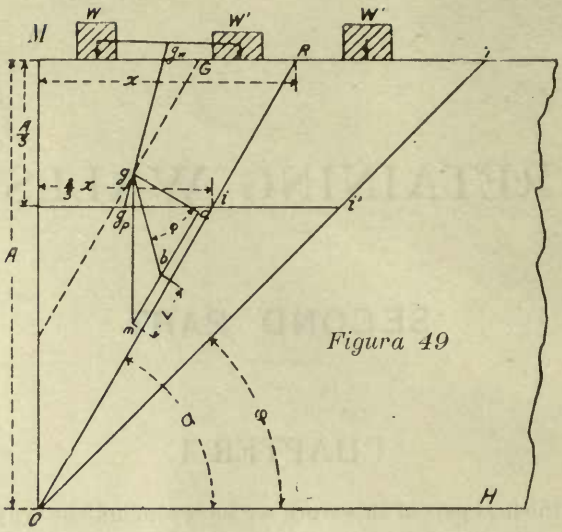


Figura 49

have to apply to it a retaining wall of sufficient size to stand the thrust produced by the unbalanced force. The direction of this force is parallel to plane α and its point of aplication is at the centre of gravity of the system (see problem N.° 13 of the coherent bodies).

It may happen, that no loads are resting on surface *M T*, but we want to give to *M O* a height greater than *A*; in this case we require also a wall to retain the earth. The plane of minimum resistance will be that one in which the difference between the unbalanced force and the shearing force ($L - \Omega S$) is at its maximum (See problem N.° 15 of the coherent bodies).

INCOHERENT BODIES

3—An incoherent body is formed by a conglomeration of independent particles, so they cannot hold themselves together at an inclination to the horizontal greater than the angle of repose, and in order that they may do so, it is necessary to place a retaining wall to keep them in position, (see problems on incoherent bodies). But the following objection incurs; in practice, we get very seldom theoretically incoherent bodies. Recently removed dump earth can be held with a more or less large vertical paramento. To be able to judge to what extent ordinary earth is coherent let us work out the following example.

4—**Example,** *Given a mass of earth, having a vertical paramento of*

10 metres in height and bounded at the top by a horizontal surface, (figure **50**) *determine:*

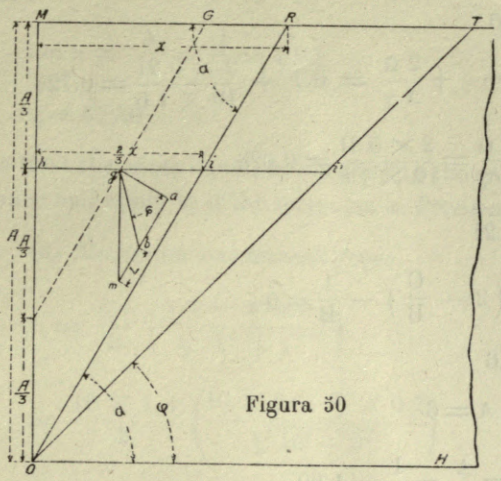


Figura 50

(a) *Maximum unbalanced force.*

(b) *Maximum horizontal component.*

(c) *Maximum vertical component.*

(d) *Angle* α.

Take 1·6 as the specific gravity of earth and 35° to be the angle of repose.

Assume the earth forming the mass, to support by its own weight, with a vertical paramento of 1 metre, i, e., *we considerer the earth to be coherent up to a height of 1 metre.*

The shearing stress on plane α is given by formula (2) of Problem N° 2 of the coherent bodies, where we had;

$$\Omega = \frac{A\,\gamma\,(1 - \sin\varphi)}{4\cos\varphi}$$

$$= \frac{1. \times 1.6\,(1 - \sin 35^\circ)}{4\cos 35^0} = 0.21$$

Substituting the above values in the formulae of Problem N.° 15 of the coherent bodies we get the four unknown required.

(a) *Maximum unbalanced force.*

$$L - \Omega S = \frac{A^2\,\gamma}{2}\left(\frac{\omega - \omega^2 B - C}{\sqrt{1 + \omega^2}}\right) = 21.440 \text{ ton.}$$

Where;

$A = 10$ m.

$\gamma = 1.6$

$$B = \tan \varphi + \frac{2\,\Omega}{A\,\gamma} = 0.7 + \frac{2 \times 0.21}{10 \times 1.6} = 0.726$$

$$C = \frac{2\,\Omega}{A\,\gamma} = \frac{2 \times 0.21}{10 \times 1.6} = 0.026$$

$\Omega = 0.21$

$$\omega^3 + \omega\left(2 - \frac{C}{B}\right) - \frac{1}{B} = 0$$

$\omega = 0{,}6$

$x = \omega\, A = 6$

$$\tan \alpha = \frac{A}{x} = \frac{1}{\omega} = 1{,}69$$

$\alpha = 59^{0}\ 24'$

(b) *Maximum Horizontal Component.*

$$(L - \Omega\, S)\cos \alpha = \frac{A^2 \gamma}{2}\left(\frac{\omega^2 - \omega^3\, B - \omega\, C}{1 + \omega^2}\right) = 11\,540 \text{ ton}$$

A, B, C, as in case (a);

$x = \omega\, A$

$$-\omega^4\, B + \omega^2 (C - 3\, B) + 2\,\omega - C = 0$$

$\omega = 0{,}74$

$$\tan \alpha = \frac{A}{x} = \frac{1}{\omega} = 1{,}353$$

$\alpha = 53^{0}\ 32'$

(c) *Maximum Vertical Component.*

$$(L - \Omega\, S)\sin \alpha = \frac{A^2 \gamma}{2}\left(\frac{\omega - \omega^2\, B - C}{1 + \omega^2}\right)$$

A, B, C, as in case (a) of and (b);

$x = A\,\omega$

$$-\omega^2 + 2\omega(C - B) + 1 = 0$$

$$\omega = 0{,}521$$

$$\tan \alpha = \frac{A}{x} = \frac{1}{\omega} = 1{,}92$$

$$\alpha = 62{,}30'$$

Now let us treat the same example considering the earth as an incoherent substance and apply to it the formulae in Problem 1, page 53.

(a) *Maximum unbalanced force.*

$$L = \frac{A\gamma}{2}\left(\frac{A\,x - x^2 \tan \varphi}{\sqrt{A^2 + x^2}}\right)$$

$$= \frac{10 \times 1.6}{2}\left(\frac{10 \times 6 \times 6^2 \times 0.7}{\sqrt{10^2 + 6^2}}\right)$$

$$= 24.000 \text{ ton.}$$

(b) *Maximum horizontal component.*

$$L \cos \alpha = \frac{A\gamma}{2}\left(\frac{A\,x^2 - x^3 \tan \varphi}{A^2 + x^2}\right)$$

$$= \frac{10 \times 1.6}{2}\left(\frac{10 \times 7.9^2 - 7.9^3 \times 0.7}{10^2 + 7.9^2}\right)$$

$$= 13.520 \text{ ton.}$$

(c) *Maximum vertical component.*

$$L \sin \alpha = \frac{A^2\gamma}{2}\left(\frac{A\,x - x^2 \tan \varphi}{A^2 + x^2}\right)$$

$$= \frac{10^2 \times 1.6}{2}\left(\frac{10 \times 5.21 - 5.21^2 \times 0.7}{10^2 + 5.21^2}\right)$$

$$= 20.880 \text{ ton.}$$

(d) *Angle* α.

For maximum unbalanced force:

$$\tan \alpha = \frac{A}{x}$$

$$= \frac{10}{6} = 1.666$$

Therefore;

$$\alpha = 59^0\ 02'$$

(e) For maximum horizontal component.

$$\tan \alpha = \frac{A}{2}$$

$$= \frac{10}{7.90} = 1.265$$

Therefore $\alpha = 51°41'$

(f) For maximum vertical component;

$$\tan \alpha = \frac{A}{x}$$

$$= \frac{10}{5.21} = 1.92$$

Therefore $\alpha = 62°30'$

To be able to compare the results obtained treating earth as coherent aad as an incoherent substance let us tabulate as follows;

Table of Comparison

MATERIAL	FORCES IN TONS					ANGLE α		
	MAX. FORCE L	HOR. COMPONENT OF L	VER COMPONENT OF L	MAX. HOR. COMP.	MAX. VERT. COMP.	MAX. FORCE L	MAX. HOR. COMP.	MAX. VERT. COMP.
Perfectly Incoherent Earth	24·000	12·232	20·560	13·520	20·880	59°02′	51°41′	62°30′
Coherent Earth	21·400	—	—	11·540	18·520	59°24′	53°32′	62°30′

Analysing the results obtained;

(1)—The maximum unbalanced force differs by 2.6 tons.

(2)—The horizontal component of the maximum unbalanced force for incoherent earth is very near equal to the maximum horizontal component for coherent earth.

(3)—The vertical component of the maximum unbalanced force for

incoherent earth is greater than the maximun vertical component for coherent earth.

(4)—Angle α in the three conditions considered corresponds very nearly in each case.

Consequently we may conclude that for designing a retaining wall to support common earth, our calculations will be limited in determining the maximum unbalanced force, and then take as the horizontal and vertical thrust on the wall, the horizontal and vertical component of the maximum unbalanced force.

LIQUIDS

5.—The phenomenon of the pressures due to liquids is more simple than that of the incoherent substances and their way of pressing is always the same (see pages 98 and following).

MASONRY WALLS

General Conditions of Stability

6.—Masonry structures are generally designed so that there is only compression stress between the blocks of which the structure is composed, the tensile strength of the material being so small, is not taken into account.

The following conditions have to be satisfied in masonry structures;

(1) *There must be no overturning.*

(2) *The shearing stress must be within the limits of the safe stress of the material.*

(3) *The maximum compressive stress must be within the safe stress for the material.*

7.—Let *a o b c* (fig. 51) represent the cross section of a masonry wall under the action of a thrust. The wall may fail by any one of the causes mentioned above due to the thrust. The ideal solution of this problem would be to find an equation comprising the three conditions of stability,

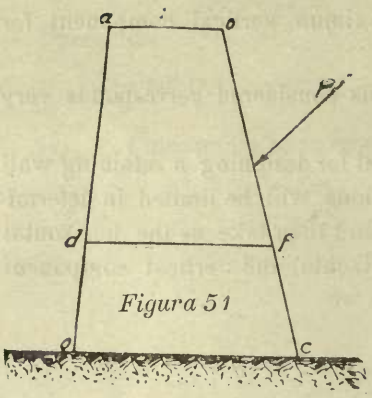


Figura 51

the graph of which would give us the cross section of the wall, but this problem if not impossible is very difficult. Being unable to design the wall using only one formula, let us proceed by parts and by trials starting from the top of the wall and working downwards, each time considering a new portion of the wall section and seeing if it adjusts to the three conditions of stability.

First Conditions of Stability

8.—Let *h M O c* (fig. **52**) represent the cross-section of a masonry wall supporting the earth pressure of prism *M O R*.

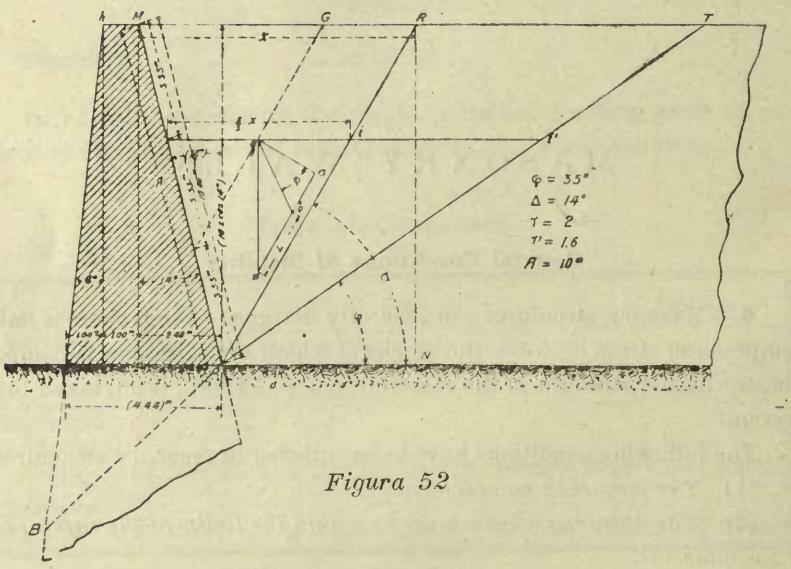


Figura 52

Let the earth have the following characteristics:

Angle of repose, $\varphi = 35°$

Specific gravity, 1·6.

Take the specific gravity of masonry, $\gamma' = 2$

The dimensions of the wall are indicated in figure.

The forces keeping the wall in equilibrium are:

The weight W of the wall acting at the c.g

The unbalanced force L acting at c.g of prism $M\ O\ R$.

The equilibriant force R, the direction and magnitude of which we require to find.

For the wall in (figure 52) we have for earth, (see problem N° 2, page 56).

Max. Hor. Comp.	19.928 tons.
Max. Vert. Comp.	30 264 tons.
Weight of wall,	52.768 tons.

Taking movements about O we have;

Unfavourable moment to stability

$$= 19.928 \times \frac{9,70}{3} = 64.434 \text{ tons - metres.}$$

Favourable moment to stability

$$= \frac{2}{3} \times 1.04 \times \frac{1.04 \times 9.70}{2} \times 2 + (1.04 + 0.50) \times 1 \times 9.70 \times 2 +$$

$$\left(1.04 + 1 + \frac{2.40}{3}\right) \times \frac{2.40 \times 9.70 \times 2}{2}$$

$$+ \left(1.04 + 1 + \frac{2.40}{3}\right) \times 30.264 = 213.229 \text{ ton meters}$$

Ratio of moments,

$$\frac{\text{Favourable}}{\text{Unfavourable}} = \frac{213.229}{64.434} = 3.32 \text{ favourable}$$

i. e., the wall is stable for overturning.

Now consider the same wall supporting water instead of earth;

For this case we have;

$$\text{Hor. Comp.} = \frac{A^2}{2} \cos^2 \triangle = \frac{10^2}{2} \cos 14° = 47 \text{ ton.}$$

$$\text{Vert. Comp.} = \frac{A^2}{2} \cos \triangle \sin \triangle = \frac{10^2}{2} \cos 14° \sin 14° = 11{\cdot}727$$

Taking moments about O, we have;

Unfavourable moment to stability

$$= 47 \text{ x } \frac{9.70}{3} = 151.810 \text{ tons - metres;}$$

Favourable moment to stability;

$$= \frac{2}{3} \times 1.04 \times \frac{1.04 \times 9.70}{2} \times 2 + (1.04 + 0.50)(1 \times 9.70 \times 2) +$$

$$\left(1.04 + 1 + \frac{2.40}{3}\right)\left(\frac{2.40 \times 9.70}{2} \times 2\right) + \left(1.04 + 1 + \frac{2.40}{3}\right) \times$$

$11.727 = 136.292$ ton. meters.

Ratio of moments;

$$\frac{\text{Unfavourable}}{\text{Favourable}} = \frac{151.810}{136.292} = 1{\cdot}111 \text{ unfavourable;}$$

So this wall when it has to support water is not stable for overturning.

Second Condition of Stability

9—The shearing stress must be within the limits of the safe stress of the material.

To study this condition of stability, let us consider the wall made up of;

(a) Separate blocks resting on plane surfaces.

(b) Only one block.

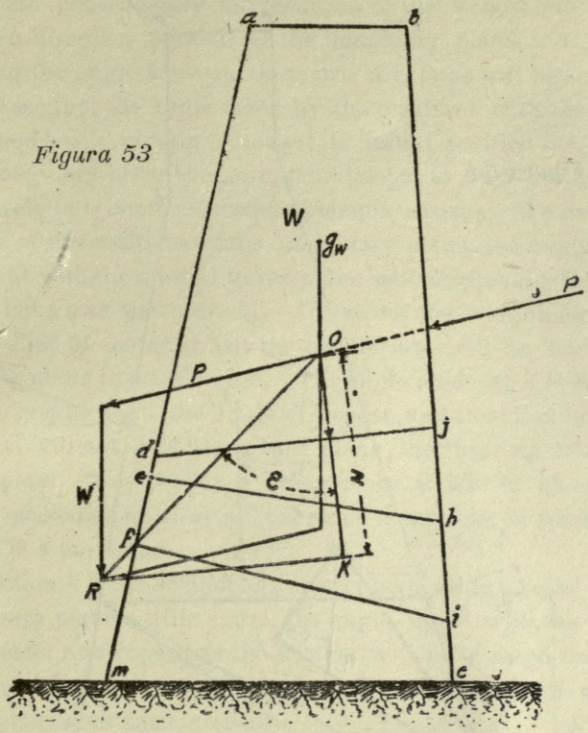
10—First Case (a) Let $a\,b\,m\,c$ (fig. 53) be the cross section of a wall and imagine that wall made up of 4 separated blocks resting on one another along the plane surfaces $d\,j$, $e\,h$, $f\,i$ and $m\,c$.

Let P be the thrust on the wall, W the weight of portion $a\,b\,d\,j$ above plane $d\,j$ and R the resultant of P and W.

In order that part of wall $a\,b\,d\,j$ will not slide along surface $d\,j$ it is necessary that angle $K\,O\,R$ made by the resultant force R with the normal to plane $j\,d$ is not greater than angle of friction φ.

The same thing applies for any other section of the wall, such as $e\,h$, $f\,i$ and for the base $m\,c$, with the only difference that angle of friction may vary if the lower part of the wall is made of different material, and for section $m\,c$ we have to take the coefficient of friction co-

Figura 53



rresponding to the material the wall is made of, and that of the foundation on which it rests.

11—Second Case (b) Let *d a b c* (fig. 54) be the cross section of a wall under the action of a thrust L_0.

In order that there may be sliding of material along any section of the wall, the resultant force of thrust and weight of wall will have to be greater than the force of friction and shearing force in the section considered.

Let us see how these forces are connected.

Take a point *h* along side *b c* of wall.

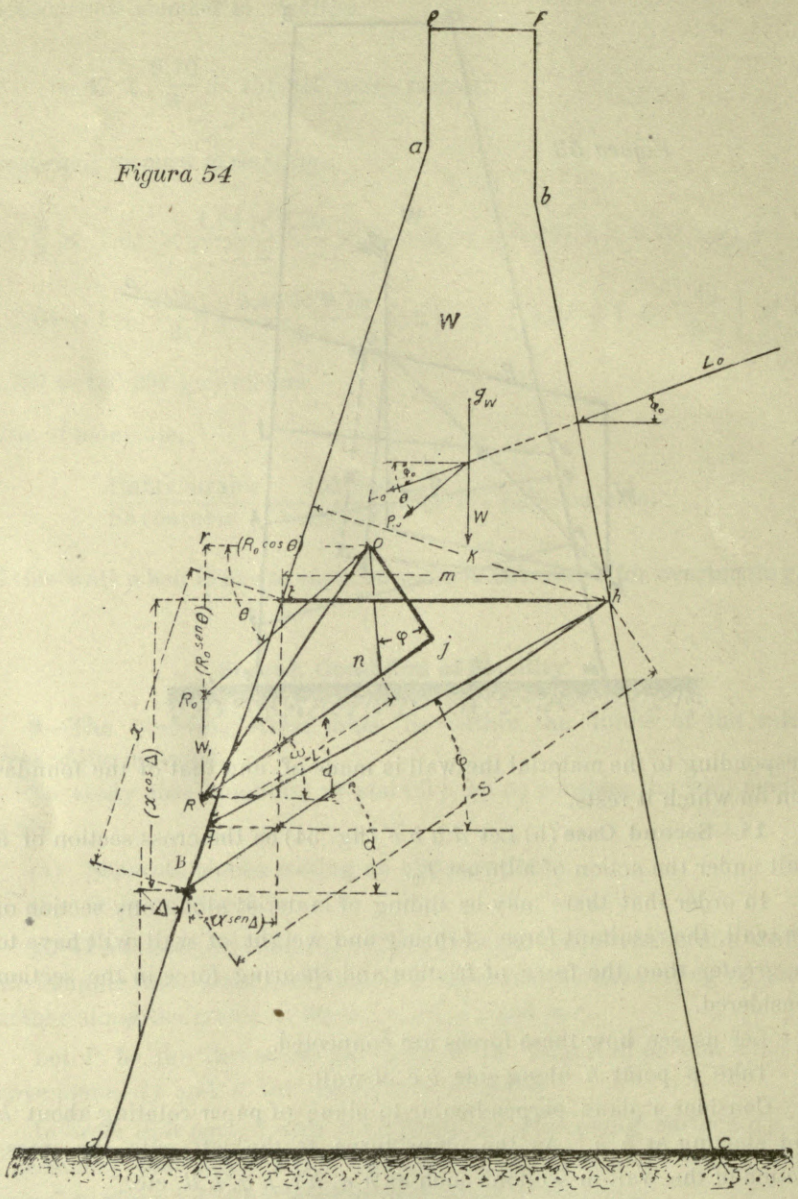
Consider a plane perpendicular to plane of paper rotating about *h* and starting at *h b*. As the plane turns to the left cuts the cross-section of the wall in sections such as *h a*, *h t*, *h q*, *h B*, etc.

In position *h b*, the pressure on plane is due to thrust L_0.

In any other position, the pressure on it will be the resultant of thrust L_0 and weight of wall resting on plane.

In position *h b* the force on L_0 is inclined to the normal at a certain positive angle, and as this plane turns, the direction of the resul-

Figura 54



tant force on plane, with the normal to plane changes from positive to negative passing throngh zero and in position $h\ t$ has certam negative value, and when plane is in a vertical position below h (not drawn) the section of the wall cut by this plane will be infinite considering the wall

indefinite and consequently the resultant of the weight and L_0 (finite) will have a direction paralell to the imaginary plane i. e., it will be vertical and the angle between these two directions will be zero.

So we see that the angle made by the resultant with the normal to cutting plane has a certain value at its initial position $h\,b$, and as we turn the imaginary plane that angle diminishes to zero and then increases negatively to become zero when plane is vertical. We can then say that in one of the positions of the imaginary plane, the angle made by the resultant with the normal to the plane will be equal to the angle of friction. Let $h\,q$ be this position. If we cut the wall along $h\,q$ there will be no slide of material, but the equilibrium will be limiting. Let us turn now plane to position $h\,B$. The angle made by $h\,B$ with the horizontal is greater than the angle of repose and therefore in order that the material will not slide along that plane, the shearing resistance comes into play. Now if sliding takes place, it will be along plane of minimum resistance to shear. This plane is that one in wich the shearing stress is a maximum.

In position $h\,B$ the section has a certain shearing stress. If we turn the imaginary plane a little more, the angle made with the horizontal with be greater and consequently friction will be less and then the new section is under a greater pressure, so the shearing force will also be greater, but the area has increased.

In position $h\,q$ the material will not slide along the plane since friction is preventing it and in the vertical position since there is no friction the material is under shearing only but the working surface is infinite.

It must consequently be an intermediate position under a maximum shear stress, this position depending on the area, the force of friction, and the external forces.

In order that this question could be treated by the calculus it is necessary that the surfaces bounding the wall could be represented by an equation. We have supposed these surfaces as planes, and can therefore be treated as equations.

Let us find the direction and magnitude of the resultant force corresponding to section, previous to maximum.

12 — Let W (fig. 55) be the weight of wall on top of plane $h\ t$ (in fig 54) and L_0 the thrust on it.

Draw triangle of forces and let R_0 be the resultant force, then R_0 is

$$(1)\quad R_0 = \sqrt{(W + L_0 \sin \alpha_0)^2 + (L_0 \cos \alpha)^2}$$
$$= \sqrt{W^2 + 2\,W\,L_0 \sin \alpha_0 + L_0^2}$$

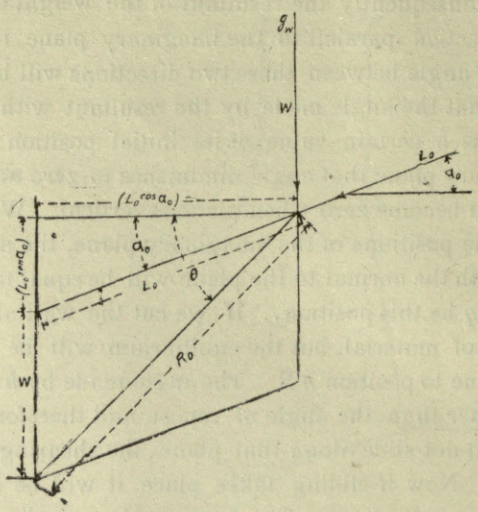


Figura 55

and the angle θ made by R_o with the horizontal is given by;

$$(2) \qquad \cos \theta = \frac{L_o \cos \alpha_0}{R_0}$$

13—Let S be the area of section $h\,t$. (fig. 54)

Then the shearing stress is $\frac{L}{S}$ where L_0 is the thrust, on the wall.

Now $\frac{L}{S}$ must be a maximum in the plane of minimum resistance.

Now let R_0 be the resultant of thrust L_0 and W, weight of wall $h\,f\,e\,t$ and θ be the inclination to the horizon.

Let also W′ represent weight of prism $ht\,B$ and R be the resultant of R_o and W′.

Resolve R in components normal and tangential to plane $h\,B$ and let $O\,J$ and $J\,R$ be those components.

Now from O draw line $O\,n$ making with $o\,j$ angle of friction φ, then $n\,R$ give us to scale the unbalanced force L.

Let $B\ t = x$.

The from figure we have;

$$R^2 = (W' + R_0 \sin \theta)^2 + (R_0 \cos \theta)^2 = W'^2 + 2\,W'\,R_0 \sin \theta + R^2_0$$

$W' = \frac{K \times \gamma}{2}$ were k height of triangle h t B.

$\frac{R_0}{\cos \omega} = \frac{R}{\cos \theta}$ where ω is angle made by R with horizontal.

$$\frac{L}{\sin (90 - \varphi - (\omega - \alpha)} = \frac{R}{\cos \varphi}$$

$$\frac{x}{\sin \alpha} = \frac{m}{\sin (90 - \triangle - \alpha)}$$

$$S^2 = (m + x \sin \triangle)^2 + (x \cos \triangle)^2 = m^2 + 2 m x \sin \triangle + x^2$$

Then eliminating W', R, ω, and α and substituting we have;

$$(1) \quad \frac{L}{S} = \frac{1}{2 \cos \varphi} \frac{x (2 R_0 \sin (\theta + \varphi + \triangle) - k \gamma m \sin \varphi) + x^2 k \gamma \cos (\triangle + \varphi) + 2 R_0 m \cos (\theta + \varphi)}{m^2 + 2 m x \sin \triangle + x^2}$$

Differentiating and simplifying we obtain;

$$\frac{d\left(\frac{L}{S}\right)}{dx} = \frac{1}{2 \cos \varphi} \frac{x^2 (k m \gamma \sin (2 \triangle + \varphi) - 2 R_0 \sin (\theta + \varphi + \triangle)) + x (2 k \gamma m^2 \cos (\triangle + \varphi) - 4 R_0 m \cos (\theta + \varphi)) + 2 R_0 m^2 \sin (\theta + \varphi - \triangle) - m^3 k \gamma \sin \varphi}{(m^2 + 2 m x \sin \triangle + x^2)^2}$$

and equating to zero for a maximum we get;

$$(2) \quad x^2 (k m \gamma \sin (2 \triangle + \varphi) - 2 R_0 \sin (\theta + \varphi + \triangle)) + x (2 k \gamma m^2 \cos (\triangle + \varphi) - 4 R_0 m \cos (\theta + \varphi)) + 2 R_0 m^2 \sin (\theta + \varphi - \triangle) - m^3 k \gamma \sin \varphi = 0$$

Now making;

$$k m \gamma \sin (2 \triangle + \varphi) - 2 R_0 \sin (\theta + \varphi + \triangle) = E$$
$$2 k \gamma m^2 \cos (\triangle + \varphi) - 4 R_0 m \cos (\theta + \varphi) = F$$
$$2 R_0 m^2 \sin (\theta + \varphi - \triangle) - m^3 k \gamma \sin \varphi = J$$

we have;

$$x^2 E + x F + J = 0$$

The root of this quadratic making the second differential coefficient negative, gives us the maximum shearing stress.

Now the angle α made by plane of minimum resistance, with horizontal is obtained from equation;

$$\cot \alpha = \frac{m + x \sin \triangle}{x \cos \triangle}$$

Therefore angle α is obtained from tables and the inclination of plane of minimum resistance to the horizontal is found.

Now in (figure 54) if starting from position $h\,t$ we want to design a wall so that all the sections of it are under the same working stress, equations (1) and (2) give us the means of doing it.

In equation (2) making $\frac{L}{S}$ equal to the shearing stress we want the material to work, and then getting rid of x in (2) we find value of angle $\triangle$ to satisfy the required condition.

The author has not succeded in giving to equation in $\triangle$ a practical form.

We may however find value of $\triangle$ by a graphic way.

GRAPHIC METHOD

14—In (fig. 56) let L_0 be the thrust on the wall and W the weight of wall above the section $h\,t$, (fig. 54) then R_0 is the resultant of both.

These three forces are constant.

Now, to same scale draw force W' as shown, to represent weight of prism $h\,t$ B, then Rg would be the resultant R, of R_0 and W'.

With Rg as diameter draw circle $g\,R\,a$ and from R draw Ra making with the horizontal angle α = inclination of section h B. From g drop line $g\,b$ making with $g\,a$ angle of friction φ, then $b\,R$ gives to scale the unbalanced force L.

Then shearing stress on section is $\Omega = \frac{L}{S}$

where S = sectional area of h B.

Do the same for other sections inclined at other angles with the horizon and compare the shearing stress for those setions with the previous one and take the maximum.

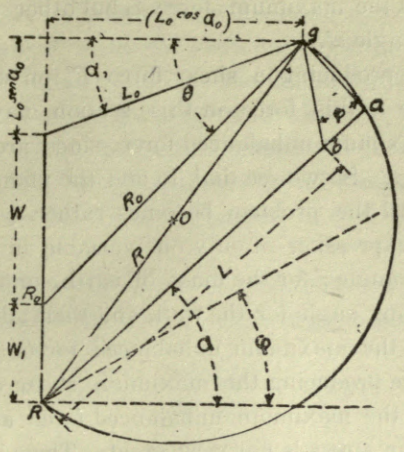


Figura 56

In the graphic construction only value of W_1 will vary, R_0 being constant.

The section on which the shearing stress is maximum gives the plane of minimum resistance and the value of α its inclination to the horizontal.

In the two methods given above to find the plane of minimum resistance on the wall, we have taken L_0 given in magnitude and direction, but this force L_0 may or may not be the thrust producing a maximum shearing stress on the wall.

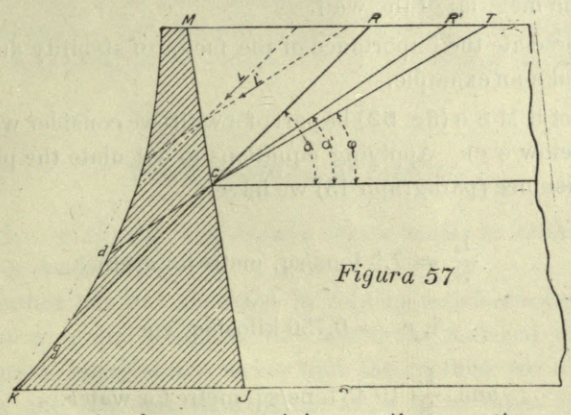


Figura 57

15—Suppose we have a retaining wall supporting earth, of the shape shown in (figure 57). Let the angle made by the maximum unbalanced force L with the horizontal be α. This force L is producing a shear force F on section $c\ g$, so the shearing stress is $\frac{F}{c\ g}$

Take now, not the maximum force L but other unbalanced force L′, inclined at an angle α'.

This force is producing a shear force F' on section $c\ d$ and the shearing stress due to this force on this section may be greater than that due to the maximum unbalanced force, since area of $c\ d$ may be less than that of $c\ g$. So we see that to find the plane of minimum resistance on the wall the problem becomes rather complex, since we have to obtain an expression of only one variable in function of L unbalanced force and angle α for the mass of earth, and unbalanced force and its corresponding angle for the wall, and then take the maximum.

So althought the maximum unbalanced force due to the earth may not be the force producing the maximum shear on the wall, the difference between the maximum unbalanced force and one producing the maximum shear stress is not very great. Their inclination to the horizontal, i. e., $\alpha - \alpha'$ is somewhere about 10^0 or less.

To appreciate this difference let us take $\alpha - \alpha' = 10^0$ and suppose the maximum thrust to be 100 tons.

The cosine of $10^0 = 0.9848078$.

The projection of 100 tons on direction α' would be $100 \times 0.9848078 = 98.48078$ tons.

Therefore error would be 1.52 % which is not very great.

So we shall not be very far wrong if we take maximum unbalanced force due to the earth, as the force causing the maximum shearing stress on the material of the wall.

To appreciate the importance of the factor of stability due to shearing let us take an example.

16—Let h M c o (fig. **52**) be part of a wall (we consider wall in figure produced below o c). Applying equations to calculate the plane of minimum resistance (paragraph 13) we have;

$$\frac{L}{S} = 7.5 \text{ tons/sq. metre for earth}$$

$$\text{i. e. } = 0.750 \text{ kilos/sq. cm.}$$

$$\text{and } \frac{L}{S} \ 10.45 \text{ tons/sq. metre for water.}$$

$$\text{i. e. } = 1.045 \text{ kilos/sq. cm.}$$

So we see that the material of the wall is working under a shearing

stress greater in case of water than when it is supporting earth. But the fact that in the case of water the shearing stress is greater than in the case of earth is not so important as the fundamental error derived from this conclusion. The weight of the wall above *o c* is 52.768 tons. (see paragraph 8). The max. horizontal component of the unbalanced force due to earth is 30.264 tons. The max. vertical component is 19.928 tons. Therefore, total vertical load on section *o c* is 52.768 + 30.264 = 83.034 tons.

The resultant force due to the normal force 83.032 and the horizontal 19.928 is inclined to the horizon at an angle ω given by

$$\tan \omega = \frac{19.928}{83.032} = 0.239$$

Therefore, $\omega = 13^{\circ}\ 27'$.

Now we have supposed the angle $\varphi = 35^{\circ}$ for masonry (see paragraph **8**).

$\tan 35^{\circ} = 0.7.$

i. e. $\omega < \varphi$.

So the wall in this condition is stable on account of friction only along the horizontal joint *o c*.

But we have shown in paragraph **16** that in plane of minimum resistance starting at *O* the material is working at a shearing stress of 0.750 kilos/sq.cm., i.e. the wall along this plane although not stable by friction only is stable on account of the resistance to shear of the material.

Third Condition of Stability

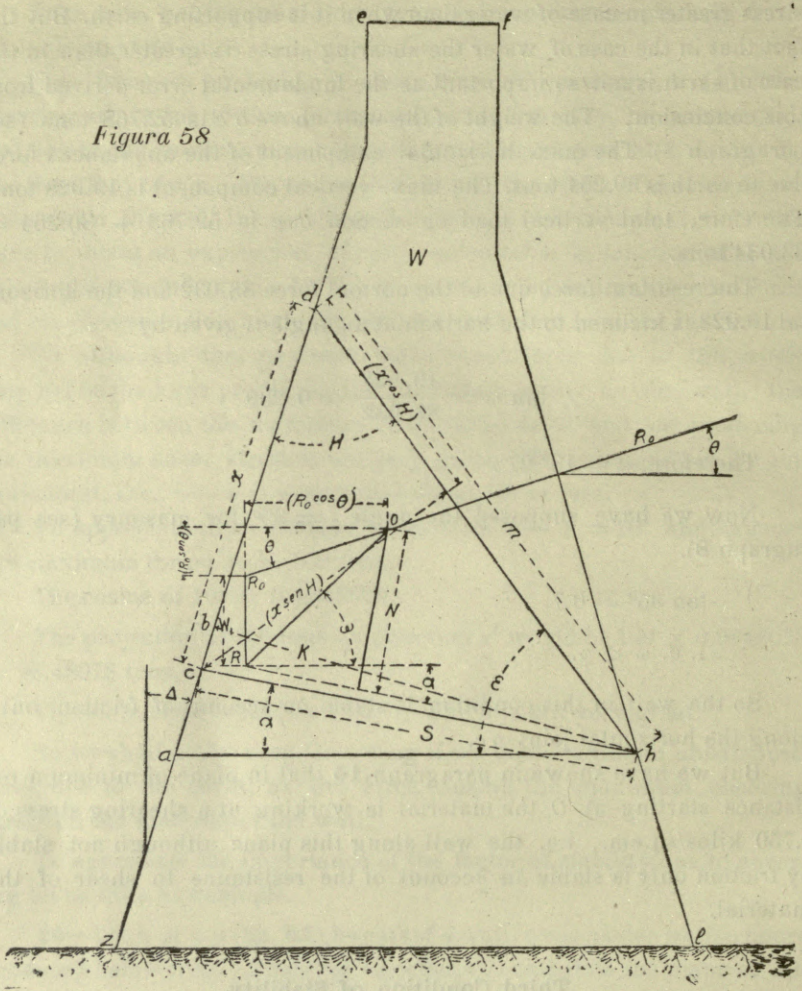
17—*The maximum compressive stress must be within the safe stress of the material.*

The author has not succeded in finding satisfactory equations to show in an easy and practical way where the material is working at the maximum compressive stress and the method we are going to describe is only an indication of the steps to be taken to get to the final result.

Let us find a section on any given wall in which the ratio of normal force to area of section is a maximum.

18—Let *l f e z* be a cross section of a wall (fig, **58**).

Figura 58



Let *h* be a fixed point on side of wall, and take this point as the axis on which an imaginary plane such as *h d* perpendicular to paper cuts the cross section as it swings to the left. Let us suppose that in plane *h d* the compressive stress is not maximum, but there is a lower plane on which ratio of normal force to area is the greatest.

Let R_o (see paragraph 12) inclined at angle θ to the horizontal be the resultant force of thrust on wall and weight of portion *h f e d*.

Find the resultant force due to weight of triangle *h d c* and force R_o and let *R* be this force inclined at an angle ω with the horizontal as shown.

Then compressive stress on section *h c* is.

$$\frac{\text{Normal Component of R}}{\text{Area of } h\,c} = \frac{N}{\text{Area } h\,c}$$

From figure we have;

(1) $R^2 = (W_1 + R_o \sin\theta)^2 + (R_o \cos\theta)^2 = W_1^2 + 2\,W_1\,R_o \sin\theta + R_o^2$

(2) $W_1 = \frac{k\,x\,\gamma}{2}$ where x is base of triangle and k its height.

(3) $\frac{R_o}{\cos\omega} = \frac{R}{\cos\theta}$

(4) $N = R \sin(\omega + \alpha)$, N being normal component of R.

(5) $\frac{m}{\sin(H + \varepsilon - \alpha)} = \frac{x}{\sin(\varepsilon - \alpha)}$

(6) $S^2 = (m - x\cos H)^2 + (x \text{ sen } H)^2 = m^2 - 2\,m\,x\cos H + x^2$

Then by elimination of R, W_1, ω, and α, we have;

Compressive stress;

$$\frac{N}{S} = \frac{1}{2}\frac{(x(k\,\gamma\,m\cos\varepsilon - 2\,R_0 \sin(\theta + \varepsilon + H)) - x^2\,k\,\gamma\cos(\varepsilon + H) + + 2\,R_o\,m\sin(\theta + \varepsilon))}{m^2 - 2\,m\,x\cos H + x^2}$$

Differentiating and simplifying;

$$\frac{d\left(\frac{N}{S}\right)}{dx} = \frac{1}{2}\Big(x^2(m\,k\,\gamma\cos(2\,H + \varepsilon) + 2\,R_o\sin(\theta + \varepsilon + H)) - - x(2\,m^2\,k\,\gamma\cos(\varepsilon + H) + 4\,R_o\,m\sin(\theta + \varepsilon)) + 2\,R_o\,m^2\sin(\varepsilon + \theta - H) + m^3\,k\,\gamma\cos\varepsilon \Big)$$

$$\frac{}{(m^2 - 2\,m\,x\cos H + x^2)^2}$$

and equating to zero for a maximum we get;

$$x^2(m\,k\,\gamma\cos(2\,H + \varepsilon) + 2\,R_o\sin(\theta + \varepsilon + H)) - x(2\,m^2\,k\,\gamma\cos(\varepsilon + H) + + 4\,R^o\,m\sin(\theta + \varepsilon)) + 2\,R_0\,m^2\sin(\varepsilon + \theta - H) + m^3\,k\,\gamma\cos\varepsilon = 0$$

Making;

$$m\ k\ \gamma \cos (2H + \varepsilon) + 2\ R_o \sin (\theta + \varepsilon + H) = E$$

$$2\ m^2\ k\ \gamma \cos (\varepsilon + H) + 4\ R_o\ m \sin (\theta + \varepsilon) = F$$

$$m^3\ k\ \gamma \cos \varepsilon + 2\ R_o\ m^2 \sin (\varepsilon + \theta - H) = J$$

we have;

$$x^2\ E - x\ F + j = 0$$

The value of this quadratic making the second differential coefficient of $\frac{N}{S}$ negative gives the maximum compressive stress on the section and the inclination of it to the horizontal is given by

$$\sin \alpha = \frac{m \sin \varepsilon - x \cos \triangle}{\sqrt{m^2 - 2\ m\ x \cos H + x^2}}$$

Graphic Method

19—If we want to design a wall so that any section on it will work at a given compressive stress the equation shown above gives us the means of doing it but in a complicated form. The best way of solving the difficulty is by the trial-graphic mothod shown in (fig. 59) where we have;

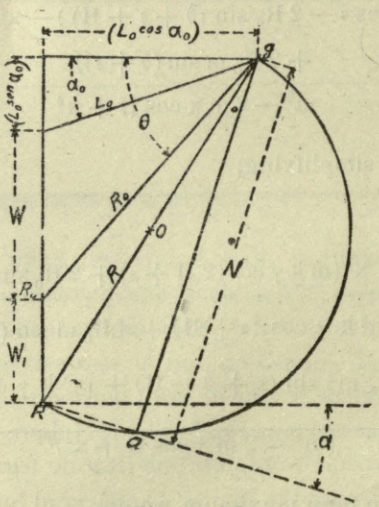


Figura 59

L_o = thrust on wall.

W = weight of wall above plane *h d*

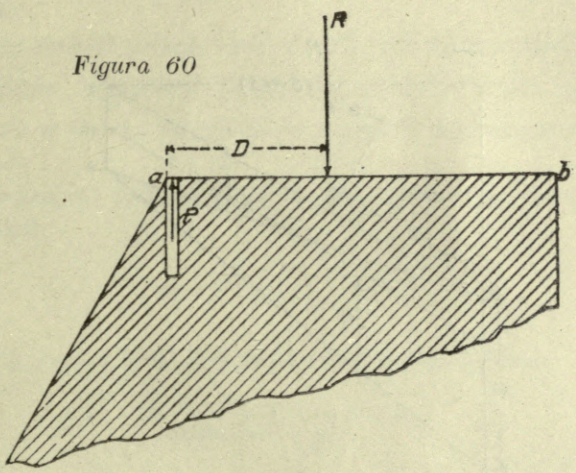
R_0 = resultant of L_0 and W
W_1 = weight of wall $h\ d\ c$
R = resultant of R_0 and W_1.
N = component of R normal to plane $h\ c$
θ = inclination of R_0 to horizontal
and α = inclination of plane $h\ c$ to horizontal.

In this way N is determined graphically and since S is known, stress on plane $h\ c$ is calculated. Do the same for other positions of plane and out of all the values obtained take the maximum. If the maximum obtained is not the one we fixed as the working stress for the design, alter angle H till we obtain the required condition.

20—We have found so far, the maximum ratio of normal force to area, what we really require is the maximum working unit stress on a section.

Let $a\ b$ (fig. 60) represent the cross-section of a masonry struc-

Figura 60



ture and let pressure R cut the cross section at a distance D from a, R being the variable pressure on it, at any direction.

Let P be the stress on a due to this thrust.

Suppose we keep constant value of R and distance D increases or diminishes; then also P increases or diminishes being its value R when $D = 0$.

i. e. P is proportional to $\frac{1}{D}$ (R benig constant)

Therefore;

P is maximum when $\frac{R}{D}$ is also maximum.

Now suppose D is constat and R varies.

If R increases, P also increases, then P is proportional to R.

Therefore,

$$P \text{ is a maximum when } \frac{R}{D} \text{ is a maximum.}$$

So in both cases maximum value of P depends on maximum ratio

$$\frac{R}{D}$$

21—Let us find the value of this stress applying the graphic method of the trapezium.

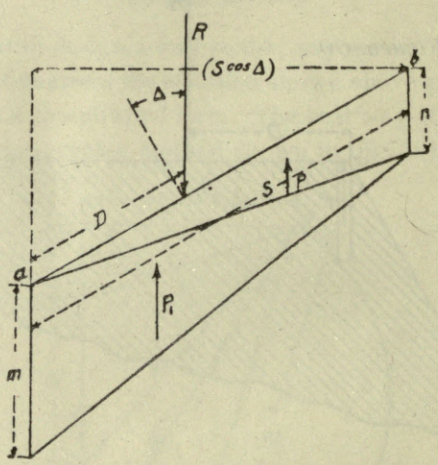


Figura 61

Let $a\ b\ m\ n$ be the diagram of pressures (figure 61) on the section considered. Let R be the resultant force calculated acting at a distance D from end a, and let S be the area of section.

Divide the trapezium in 2 triangles as shown and let us apply forces P and P_1 at their c.gs and equal to the pressures on the triangles.

Since these three forces are in equilibrium their moment about point a is zero.

Then by taking moments we have;

$$R\,D\cos\triangle = \frac{1}{3}\,P_1\,S\cos\triangle + \frac{2}{3}\,P\,S\cos\triangle$$

Also $$R = \frac{m+n}{2}\,S\cos\triangle$$

$$P_1 = \frac{m\,S\cos\triangle}{2}$$

$$P = \frac{n\,S\cos\triangle}{2}$$

and by substitution we get;

$$m = \frac{4\,R\,S - 6\,R\,D}{S^2\cos\triangle}$$

22—In practice we may require to find the maximum stress on any section.

Now in several planes such as $h\,c$ find the resultant force on it, take the normal component N and the plane for which the ratio $\frac{N}{D}$ is maximum is the one en which the stress is maximum at the compression side.

In the general method given to find stress P knowing R and S we found that;

$$m = \frac{4\,R\,S - 6\,R\,D}{S^2\cos\triangle}$$

then making $\triangle = 0$ force R would be the normal N, then;

$$m = \frac{4\,N\,S - 6\,N\,D}{S^2}$$

MASONRY DAMS

23—In the design of masonry dams for retaining water we have to take into account more conditions than for the design of a retaining wall. In this sort of work, being as a rule of great sizes and considering that its failure may give rise to a great disaster to the population downstreams, its stability must be a factor of the greatest importance and its cost an item that cannot be overlooked.

We may then say that in dam work the chief conditions for design

are safety combined with cheapness, the shape of the dam being a factor of second order although of great importance.

To calculate the section of a dam we have to assume the width at the top and this width varies according to the traffic of the top of the dam, the impact of the block of ices if they are likely to be formed and perhaps some other local condition. Mr. Wegmann, a great authority on the subject, recomends a width at the top of 20 feet, i. e., about 6 metres. So taking a width of 6 metres at the top, let us find the section of a dam to retain water.

Let *g h* (fig. 62) be the width at the top and *g h i d* be the upper por-

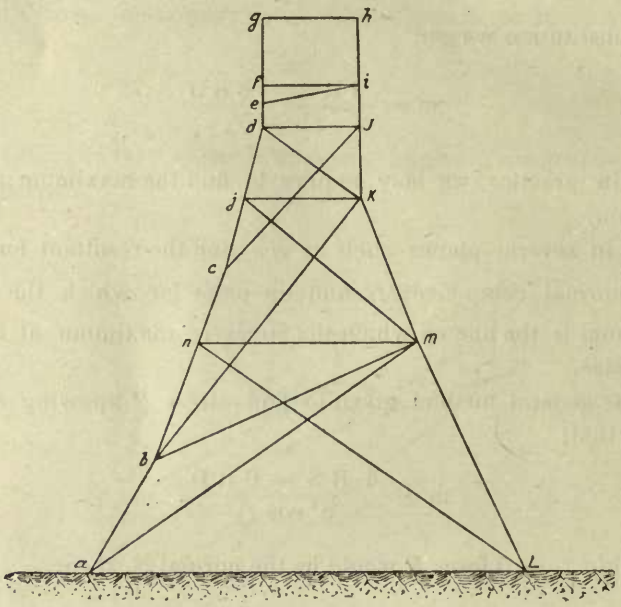


Figura 62

tion with vertical sides, comprising the three conditions of stability for overturning, shearing and compressive stress. The way of calculating the section is the same as for a retaining wall with the only difference that for the condition of overturning the ratio of the favourable moment of stability to the unfavourable must be within 2 and 3. With reference to the other two conditions of stability a greater factor of safety must be taken for dams than for retaining walls.

There is a difference in the shape for retaining walls and for dams and this is that in walls, the shape of the front part is generally fixed beforehand whilst in dam work this is not the case; but this difference in shape does not interfere with the calculations and the method of

designing both is practically the same. The forces acting on a wall may be considered constant in magnitude, point of application and sense, whilst in a dam vary according to whether the reservoir is empty or full. If the reservoir is empty the only force trying to overturn the wall is the wind pressure, but as this pressure is so small compared with the water pressure when it is full, it is not considered in the design.

We have obtained the rectangular section *g h j d* for the three conditions of stability and to proceed with the design we calculate the inclination $\triangle$ that the back of the dam must make with the vertical in order that material of dam should work at a given shearing stress. So having fixed the working shearing stress, angle is obtained from equation (1) and (2) paragraph 13.... or by the graphic method given in paragraph 14. Suppose now that $\triangle$ has been determined and that plane of minimum resistance is *j c*. Next calculate in the same way the plane of minimum resistance *d k* for the reservoir empty and draw the horizontal *k j*. Now with *k j* as the starting base we determine in the same way as before the planes of minimum recistance *k b* and *j m*, and also *m a* and n L starting at *m n*.

In this way the shape of the section of the dam, so that the condition of shearing stress may be sastified, has been determined. Our next step is to check for compression by the graphic method of paragraph 20 and find the maximum compressive stress. If the compresive stress obtained is not within the limits allowed by the material the width of the section is increased on the side that is working at the greatest compressive stress, till it is safe.

In the design of a section for a dam the working shearing stress on the material is the factor to consider chiefly. The planes of minimum resistance make with the horizontal a negative angle, i. e. are always below the horizontal, while for the compressive stress the planes are making positive angles with the horizontal. The overturning of a section given by taking moments is a factor of safety involving force and distance without taking into account the way in which the force is applied or balanced and consequently cannot be taken as a condition to calculate the shape of a dam section.

Being a factor of safety of great importance for stability it is advisasable to take as the ratio of moments 3 or as near as 3 as possible and if the section calculated as good for compression and shear does not satisfy the last condition the width of the section is increased to the safe limit.

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